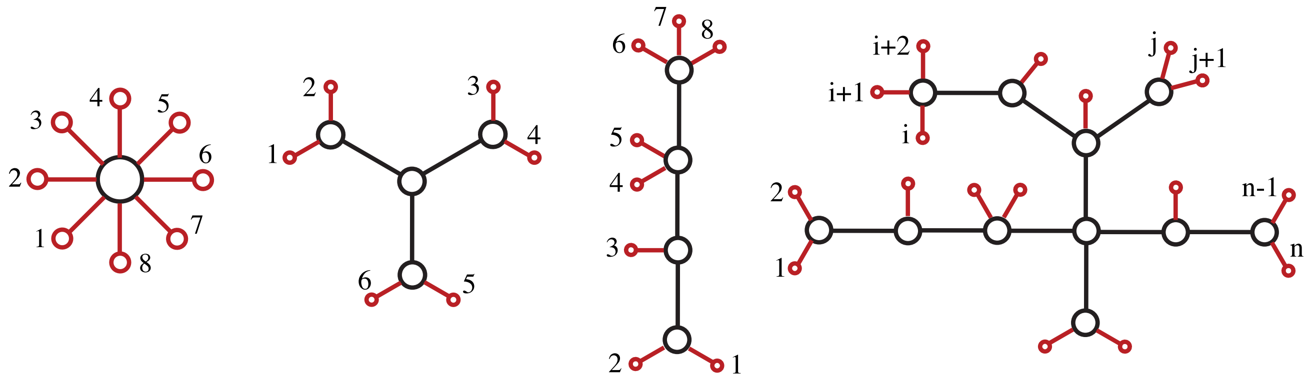

Lecture Course: Faculdade de Ciências, Universidade de Lisboa, February 2023

"Moduli of n points on the projective line"

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This mini-course addresses Master and Doctoral students as well as Postdoc and Senior researchers in mathematics.

Objective: The problem of constructing normal forms and moduli spaces for various geometric objects goes back (at least, and among many others) to the Italian geometers (Enriques, Chisini, Severi, ...). A highlight was reached in the 1960 and 70es when Deligne, Mumford and Knudsen investigated and constructed the moduli space of stable curves of genus g. These spectacular works had a huge impact, though the techniques from algebraic geometry they applied were quite challenging. In the course, we wish to offer a gentle and hopefully fascinating introduction to these results, restricting always to curves of genus zero, that is, transversal unions of projective lines $\mathbb{P}^1$. This case is already a rich source of ideas and methods.

Contents: We start by discussing the concept of (coarse and fine) moduli spaces, universal families and the philosophical background thereof: why is it natural to study such questions, and why the given axiomatic framework is the correct one? Once we have become familiar with these foundations (seeing many examples on the way), we will concentrate on n points in $\mathbb{P}^1$ and the action of PGL_2 on them by Möbius transformations. This is part of classical projective geometry and very beautiful. As long as the n points are pairwise distinct, things are easy, and a moduli space is easily constructed. Things become tricky as the points start to move and thus become closer to each other until they collide and coalesce. What are the limiting configurations of the points one has to expect in this variation? This question has a long history - Grothendieck proposed in SGA1 a convincing answer: n-pointed stable curves.

We will take at the beginning a different approach by proposing an alternative version of limit. Namely, we embed the space of n distinct points in a large projective space and then take limits therein via the Zariski-closure. This opens the door to the theory of phylogenetic trees: they are certain finite graphs with leaves and inner vertices as a tree in a forest. Their geometric combinatorics will become the guiding principle to design many proofs for our moduli spaces. Working with phylogenetic trees can be a very pleasing occupation, we will draw, glue, cut and compose these trees and thus get surprising constructions and insights.

At that point a miracle happens: The stable curves of Grothendieck, Deligne, Mumford, Knudsen pop up on their own. We don't even have to define them - they are just there. So the circle closes up, and our journey is now able to reprove many of the classical results in an easy going and appealing manner.

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