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Abstract

We study, in the context of the Unit-Demand Vehicle Routing Problem, the relationship between the linear programming relaxation of a single-commodity flow model and the linear programming relaxation of a pure time-dependent formulation that is closely related to a formulation of the travelling salesman problem. We show that the time-dependent formulation implies a new large class of upper bounding and lower bounding flow constraints that are not implied by the linear programming of the single commodity flow model. Using the time-dependent formulation we also show how to generate two new sets of inequalities in the space of the design variables: one set contains inequalities that are related to the well-known multistar constraints while the other contains inequalities that are related to double jump inequalities. Computational results show that the time dependent formulation can be used to easily solve instances with up to 80 nodes, when the vehicle capacity is reasonably small.

Keywords: Vehicle Routing Problem, Time-dependent Formulations

1. Introduction

The unit-demand Capacitated Vehicle Routing Problem (CVRP) is defined on a given directed graph G = (V,A) with a node set V = {1,…,n} and an arc set A with an integer weight (cost) c_a associated with each arc a of A, as well as a given natural number Q. The problem seeks a minimum cost set of routes originating and terminating at the depot (we assume that node 1 is the depot) with each node in V\{1}

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visited exactly once and each route containing at most Q nodes (plus the depot). The CVRP is closely related to delivery type problems and appears in a large number of practical situations concerning the distribution of commodities. The book by Toth and Vigo (2002) provides surveys on the problem, including variants and solution techniques. Papers by Araque et al. (1994), Lysgaard, Letchford and Eglese (2004) and Fukasawa et al. (2006) discuss the most successful algorithms for solving this problem as well as general demand cases. The Araque et al method is a cutting plane method that views the CVRP as an equivalent path partitioning problem (PPP) and uses routines for separating several facet defining inequalities of the PPP polytope, such as generalized subtour elimination constraints, and large, intermediate and small multistars that were identified and introduced in the paper by Araque, Hall and Magnanti (1990). The Lysgaard, Letchford and Eglese method is a cutting plane method using rather efficient routines for separating well-known inequalities for the CVRP such as generalized subtour elimination constraints, multistars, knapsack multistars, framed capacity inequalities, combs inequalities, and hypotours inequalities. The method described in Fukusawa et al. goes a step further by efficiently combining cutting planes with column generation. Several authors have studied the polyhedral structure of the CVRP, among them, Laporte and Nobert (1984), Campos, Corberan and Mota (1991), Araque, Hall and Magnanti (1990), Araque (1990), Augerat (1995) and Letchford, Eglese and Lysgaard (2002). The paper by Letchford and Salazar-Gonzalez (2006) provides a recent comparison of the linear programming relaxation of several formulations presented in the literature and the recent paper by Godinho, Gouveia and Magnanti (2007) presents and compares the linear programming relaxation of several so-called multicommodity flow time-dependent formulations.

In this paper we study the relationship between the linear programming relaxation of a pure time-dependent formulation (that can be seen as modified version of the well-known Picard and Queyranne (1978) formulation for the TSP) and the linear programming relaxation of a well known single-commodity flow model due to Gavish and Graves (1978). In particular, we show that the time-dependent formulation implies a new large class of upper bounding and lower bounding flow constraints that are not implied by the linear programming of the single commodity flow model. By using some well-known techniques for generating inequalities in the space of the design variables from the linear programming feasible set of single commodity flow models (see, for instance, Gouveia (1995) and, Letchford and Salazar-Gonzalez (2006)) we show how to generate new inequalities that are related to the well-known multistar constraints (see Araque, Hall and Magnanti (1990)). We also show how to generate a conditional version of the double jump inequalities proposed in Godinho, Gouveia and Magnanti (2007). Finally, we present some computational results showing that an enhanced version of the time dependent formulation can be used to easily solve instances with up to 80 nodes, when Q is reasonably small.

In Section 2 we review a well-known single commodity flow formulation for the CVRP and in Section 3 we present the modified Picard and Queyranne formulation together with a set of valid inequalities in the space of the time-dependent variables that improve the linear programming relaxation of the modified formulation. In Section 4 we state a result relating the linear programming relaxations of these two formulations. Section 5 describes some new upper bound inequalities and new lower bound flow inequalities in the space of the design and flow variables and Section 6 describes new conditional double-jump inequalities and multistar-like inequalities that are defined in the space of the design variables. Section 7 presents computational results comparing the linear programming bounds of the formulations discussed in the paper.

2. A Generic Formulation and the Single-Commodity Formulation for the CVRP

Throughout this paper, for any formulation P, F(P) denotes its set of feasible solutions, v(P) the value of an optimal solution, and P_L its linear programming relaxation. If g is any quantity defined on the arcs (i,j), we let $g(A,B)$ denote $\sum_{i\in A}\sum_{j\in B} g_{ij}$. When A = B, we will simply use the notation $g(A)$ and when A (or B) is a single node set, for instance A = {i}, we will use $g(i,B)$. In particular, we will use this notation when referring to binary arc variables x_{ij} associated with arc inclusion.

Consider the following generic formulation for the CVRP:

$$\begin{aligned}
&\textit{minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} \\
&\textit{subject to} \quad x_{ij} \in \textit{Assign} \quad (i,j) \in A \\
&\qquad\qquad \{(i,j):\ x_{ij} = 1 \textit{ does not contain routes with more than } Q \textit{ nodes}\}
\end{aligned}$$

with Assign denoting the feasible set of the well-known assignment relaxation arising in formulations for the problem:

$$\begin{aligned}
&x(V,j) = 1 \quad j \in V \setminus \{1\} \\
&x(i,V) = 1 \quad i \in V \setminus \{1\} \\
&x_{ij} \in \{0,1\} \quad (i,j) \in A.
\end{aligned}$$

There are several ways to model the implicit route constraints. We obtain probably the most well-known formulation for the CVRP by modelling them as the standard generalized subtour elimination constraints $x(S) \le |S| - \lceil |S|/Q \rceil$ for all S $\subseteq$ V\{1} (see, for instance, Araque, Hall and Magnanti (1990)) that limit the size of each route by guaranteeing a sufficient number of arcs incoming into each subset of nodes and so ensuring that the nodes in the subset can be split into feasible routes. Letchford and Salazar-Gonzalez (2006) summarize and discuss several variants of these inequalities from the literature.

An alternative modelling approach is to use extra variables to express the implicit constraints (see, for instance, Letchford and Salazar-Gonzalez (2006) and Godinho, Gouveia and Magnanti (2007)). Probably, the most well-known such formulation for the CVRP is the following single commodity flow formulation (see Gavish and Graves (1978)) that uses additional flow variables f_{ij} indicating the amount of flow on arc (i,j) (assuming that the depot, node 1, sends one unit of flow to every other node):

$$
\begin{aligned}
&\textit{minimize} \quad \sum_{(i,j)\in A} c_{ij}x_{ij} \\
&\textit{subject to} \quad x_{ij} \in \textit{Assign} \\
&\qquad f(V,j) - f(j, V\setminus\{1\}) = 1 \qquad && j \in V\setminus\{1\} \\
&\qquad f(1, V\setminus\{1\} = n-1 \\
&\qquad f_{ij} \le (Q-1)x_{ij} && (i,j)\in A,\ i,j \ne 1 \\
&\qquad f_{1j} \le Qx_{1j} && j \in V\setminus\{1\} \\
&\qquad f_{ij} \ge x_{ij} && (i,j)\in A,\ j \ne 1 \\
&\qquad f_{ij} \ge 0 && (i,j)\in A.
\end{aligned}
$$

We assume the flow on each arc entering or leaving the depot node is equal to zero (that is $f_{j1} = 0$ for all j $\in$ V\{1}). The linking constraints $f_{1j} \le Qx_{1j}$ guarantee that the flow on each arc leaving the root does not exceed Q. This restriction, together with the flow conservation constraints and the remaining linking constraints, guarantees that each route cannot contain more than Q client nodes. Gouveia (1995) has shown that the projection of the feasible set of the linear programming relaxation of the single commodity formulation is completely described by the equalities in *Assign*, plus nonnegativity constraints imposed upon the x_{ij} variables and the so-called multistar constraints (see Section 6) that are known to be facet defining for the CVRP polytope (see Araque, Hall and Magnanti (1990)). Araque, Hall and Magnanti (1990) and Letchford (2002) have also proposed variations of these constraints. Letchford and Salazar-Gonzalez (2005) discuss similar classes of inequalities that result from a similar projection result for several other variants of the CVRP. In Section 6 we describe variations of these constraints which, as far as we know, appear to be new.

3. The Modified Picard and Queyranne Formulation

The well-known Picard and Queyranne formulation for the TSP (see Picard and Queyranne (1978)) is easily modified for the CVRP (as far as we know, no one has previously given explicit reference to this formulation for the CVRP). As in the Picard and Queyranne formulation for the TSP, we use an expanded layered graph. Two copies of node 1 serve as the source and the destination nodes. A node j^h (h=1,…,Q), represents a copy of node j (j = 2,…,n) in layer h. The network contains: i) arcs from the source version of node 1 to any node in the levels 1 to Q, ii) arcs from nodes in level h to nodes in level h+1 (h = 1,…,Q-1), but we do not allow arcs linking copies of the same original node and, iii) arcs from nodes in level Q to the destination version of node 1. We will also say that an arc directed into a node 1 at level h, has position h).

As an illustration, Figure 3.1 shows the layered graph representation of the feasible CVRP solution depicted also in Figure 3.1, an instance with n=7 and Q=3 (labels associated with the arcs correspond to the value of the flow in the arc, that is the number of nodes still to be visited in the route).

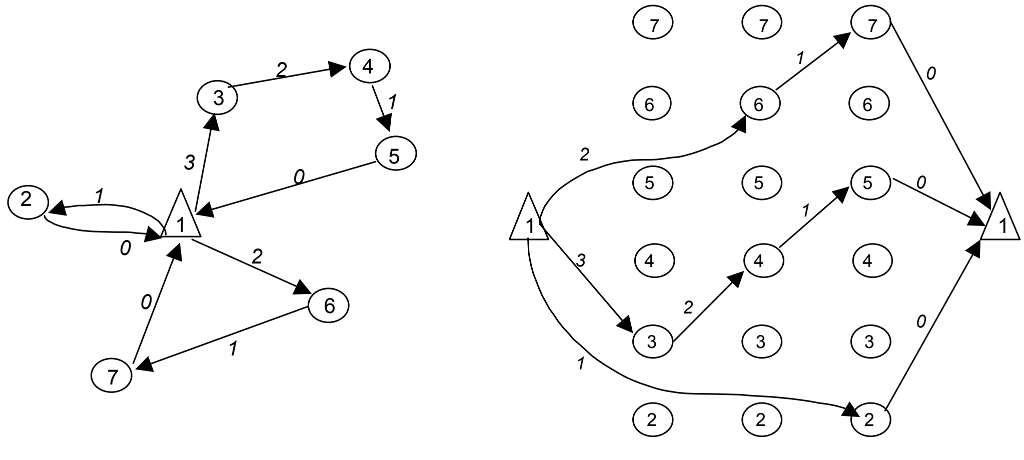

Figure 3.1 – A feasible solution for a CVRP instance with n = 7 and Q = 3 and the corresponding layered graph.

The main difference between this network and the layered graph construct for the TSP is that now we i) allow arcs leaving the source node 1 to nodes in layers h = 2,…,Q (to allow for vehicle routes with fewer than Q nodes), and ii) allow more than one path linking the source and destination versions of node 1 (to represent several vehicle routes). We obtain the Modified Picard and Queyranne formulation, MPQ, by replacing the implicit part of the generic formulation with the following system:

$$
\begin{aligned}
& z_{1j}^{1} = \sum_{i \in V\setminus\{1\}} z_{ji}^{2} && j = 2,\ldots,n \\
& \sum_{i \in V} z_{ij}^{h} = \sum_{i \in V\setminus\{1\}} z_{ji}^{h+1} && j = 2,\ldots,n \text{ and } h = 2,\ldots,Q-1 \\
& \sum_{i \in V} z_{ij}^{Q} = z_{j1}^{Q+1} && j = 2,\ldots,n \\
& x_{1j} = \sum_{h=1,\ldots,Q} z_{1j}^{h} && j = 2,\ldots.,n \\
& x_{ij} = \sum_{h=2,\ldots,Q} z_{ij}^{h} && i,j = 2,\ldots.,n \\
& x_{j1} = z_{j1}^{Q+1} && j = 2,\ldots,n \\
& z_{ij}^{h} \in \{0,1\} && (1,j) \in A \text{ and } h = 1,\ldots.,Q \\
& && \text{or } (i,j) \in A,\ i,j \neq 1 \text{ and } h = 2,\ldots,Q \\
& && \text{or } (i,1) \in A \text{ and } h = Q+1.
\end{aligned}
$$

In this model, the variable z_{ij}^{h} indicates that arc (i,j) is in a path that contains Q-h+1 nodes after the arc (including node j, but not node 1). The equality constraints imposed upon the z variables simply define a network flow system in this layered graph whose solution (in integer variables) are paths (corresponding to routes in the original graph) from the source to the destination versions of node 1. These constraints permit each path to visit several copies of the same original node. However, in the overall problem, the constraints linking the z_{ij}^{h} with the x_{ij} variables and the assignment constraints in the x_{ij} variables rule out that situation. Note that the equality constraints relating the x_{ij} and z_{ij}^{h} variables permit us to rewrite the MPQ formulation with only the z_{ij}^{h} variables.

The existence of these two models, suggest three questions:

Q1. What is the relationship between the linear programming relaxation of the two formulations, SCF and MPQ?

In Section 4 we will show that the linear programming relaxation of MPQ is at least as good as the linear programming relaxation of SCF. This result suggests two other questions:

Q2. What are the improvements, if any, of the linear programming bounds of MPQ over the linear programming bounds of SCF?

Q3. What inequalities are implied by the linear programming relaxation of MPQ that are not redundant in the linear programming relaxation of SCF?

Section 9 gives some computational results showing what we can gain by using the MPQ model instead of the SCF model when solving CVRP instances and Sections 5 and 6 give a partial answer to question Q3.

Before closing this section we introduce one set of inequalities in the space of the z_{ij}^h variables that considerably improves the linear programming bound of the MPQ formulation. The new inequalities state that if arc (j,i) is in position h+1, then some arc other than (i,j) is in position h

$$\sum_{k \in V; k \neq i} z_{kj}^h \geq z_{ji}^{h+1} \quad \text{for all } i, j = 2,...,n \text{ and } h = 2,...,Q-1. \tag{3.1}$$

As far as we know, these inequalities were first proposed in Gouveia (1999) for tree problems and were later on used as valid inequalities to enhance hop-indexed formulations for the capacitated minimum spanning tree problem (see Gouveia and Martins (1999,2000). Recently, Cordeau, Costa and Laporte (2006) have used use a formulation involving inequalities (3.1) for a variation of the Steiner Tree Problem with revenues, budget and hop constraints.

Note that by adding z_{ij}^h to each side of this inequality, by using the constraints from MPQ and canceling equal terms, we can rewrite these constraints in symmetrical form (as compared to (3.1)) as follows:

$$\sum_{k \in V \setminus \{1\}; k \neq i} z_{jk}^{h+1} \geq z_{ij}^h \quad \text{for all } i, j = 2,...,n \text{ and } h = 2,...,Q-1.$$

We let MPQ* denote the MPQ model augmented with inequalities (3.1) and we have

Proposition 3.1: $v(MPQ^*_L) \geq v(MPQ_L)$.

As noted before, it is quite easy to find instances for which the previous inequality is strict. Clearly, questions similar to questions Q2 and Q3 also apply for the MPQ* model in relation with the models MPQ and SCF.

4. Relating the linear programming relaxations of MPQ and SCF

For the next result, we need to add to MPQ the following definitional constraints that link the z_{ij}^h variables with the flow variables f_{ij} of the SCF model. This addition does not alter its linear programming bound (for simplicity, we will use the same designation MPQ to refer to this augmented model)

$$f_{1j} = \sum_{h=1,...,Q} (Q-h+1) z_{1j}^h \quad j = 2,....,n \tag{4.1}$$

$$f_{ij} = \sum_{h=2,..,Q} (Q-h+1) z_{ij}^h \quad (i,j) \in A;\ i,j \neq 1 \tag{4.2}$$

$$f_{j1} = 0 z_{j1}^{Q+1} \quad j = 2,...,n. \tag{4.3}$$

The following result and proof are a simple adaptation of a similar result and proof given in Gouveia and Voss (1995) for the Traveling Salesman Problem (for simplicity we skip the proof and refer the reader to Godinho et al. (2007)).

Proposition 4.1: The projection of F(MPQ_L) into the space of the variables x_{ij} and f_{ij} is contained in F(SCF_L).

Corollary: v(MPQ_L) ≥ v(SCF_L).

5. Some Inequalities Implied by MPQ (and MPQ*) in the x_{ij} and f_{ij} Space

As noted in Section 4 the linear programming relaxation of the MPQ formulation is at least as strong as the linear programming relaxation of the SCF formulation. Some computational results in Section 9 show that, in general, the linear programming relaxation of MPQ strictly dominates the linear programming relaxation of SCF (see also Figure 5.1 to follow). In this section we describe several inequalities in the space of the variables x_{ij} and f_{ij} that are implied by the linear programming relaxation of MPQ and that are not implied by the linear programming relaxation of the SCF formulation. As motivating one such set of constraints, we note that we could have defined a valid version of the SCF model with a weaker linear programming relaxation by using the weaker linking constraints:

$$\begin{aligned} &f_{ij} \leq Q x_{ij} \qquad && (i,j) \in A, \\ &f_{ij} \geq 0 && (i,j) \in A. \end{aligned}$$

The stronger set included earlier in the SCF formulation models the fact that the flow in any "internal" arc with $x_{ij} = 1$ cannot exceed Q-1 (since the first node in any route absorbs one unit of flow) and cannot be less than 1 (since only the arc returning to the depot would have flow equal to zero). We can generalize this concept by introducing a variation of the stronger constraints that reflects the maximum or minimum flow in arcs in positions two away (after or before) node 1. The validity of the following constraints is easy to establish:

$$\begin{aligned} &f_{ij} \leq (Q-2)x_{ij} + x_{1i} \qquad && (i,j) \in A,\ i,j \neq 1 \\ &f_{ij} \geq 2x_{ij} - x_{j1} && (i,j) \in A,\ j \neq 1. \end{aligned} \qquad (5.1a/b)$$

Note that these constraints do not imply the previous set. However, we can add them to the single commodity flow model to tighten the linear programming relaxation. Figure 5.1 depicts a feasible solution for the linear programming relaxation of SCF for an instance with n = 6 and Q = 3. On the left, the labels associated with the arcs correspond to values (those greater than zero) of the x_{ij} variables and on the right the labels associated with the arcs correspond to values (again those greater than zero) of the flow variables.

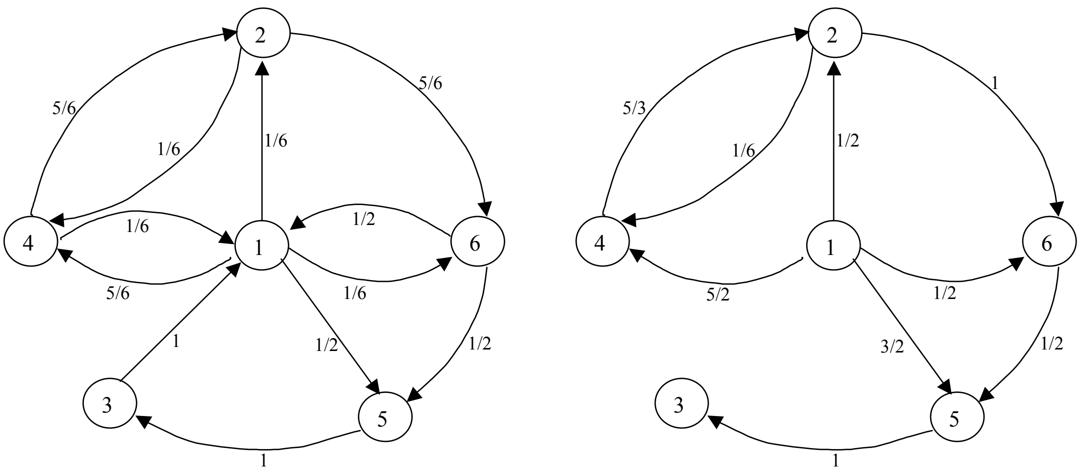

Figure 5.1 – A feasible solution for the SCF_L model.

It is easy to see that the solution shown in Figure 5.1 violates the constraint $f_{26} \geq 2x_{26} - x_{61}$. Note that increasing the flow in this arc would violate the upper bound constraint $f_{26} \leq x_{26} + x_{12}$.

There are two ways we would like to generalize the constraints (5.1a/b): (i) bound the flow in arcs that are more than 2 arcs away from the depot, and (ii) consider constraints for arc sets instead of a single arc (i,j).

5.1 Generalizing Inequalities (5.1a/b) for Arcs That Are More Than 2 Arcs Away From the Depot

In any feasible solution to the problem, we will say that an arc is at least k arcs away from the depot if it is the $(k+1)^{st}$ or later arc on a path (route) from the depot. The constraints (5.1a/b) bound the flow on arcs that are at least two arcs away from the depot. To obtain analogous bounds for arcs that are at least k arcs away from the depot, for simplicity, we consider only a generalization of the upper bound constraint (5.1a). We begin by writing a constraint of the form

$$f_{ij} \leq (Q-1-k)x_{ij} + Exp(x,k) \qquad (i,j) \in A,\ i,j \neq 1, k = 1,...,Q-3$$

with $Exp(x,k)$ denoting a linear term in the x variables that depends on k and should equal p (p ≤ k) if the path from node 1 to node i contains k-p+1 arcs. To motivate this expression, we consider a set of inequalities for k = 2, but involving quadratic terms.

$$f_{ij} \le (Q-3)x_{ij} + 2x_{1i} + \sum_{k \in V\setminus\{1,i\}} (x_{1k} \times x_{ki}) \qquad (i,j) \in A,\ i,j \ne 1 .$$

To show that this inequality is valid, we note the following: i) if the path from node 1 to node i contains three or more arcs, then the flow in arc (i,j) is at most Q-3, the two terms on the right are zero, and the inequality is valid; ii) if arc (1,i) is in the solution, then the middle term is equal to 2, the right most term is equal to zero and again, the inequality is valid; iii) if the path from node 1 to node i contains two arcs, then one of the terms $x_{1k} \times x_{ki}$ equals 1, the middle term equals zero, and since the flow on arc (i,j) is at most Q-2, the inequality is valid. Using the fact that $x_{1k} \times x_{ki} \le x_{1k}$ and $x_{1k} \times x_{ki} \le x_{ki}$ for each k in V\{1,i}, we can generate the following set of linear inequalities from the previous nonlinear one

$$f_{ij} \le (Q-3)x_{ij} + 2x_{1i} + x(1,P) + x(V\setminus(P \cup \{1\}),i) \qquad (i,j) \in A,\ i,j \ne 1, P \subseteq V\setminus\{1,i\} . \tag{5.2a}$$

Notice that this inequality captures condition iii) from the previous argument since for any node set P, the node in the middle of the 2-path joining node 1 to node i either is in the set P (and then, the second term from the right is equal to 1) or is not in P (and then, the first term from the right equals 1). It is easy to see how to strengthen the inequalities (5.2a) by excluding node j from P. Examples can be found showing that these inequalities are not implied by the linear programming relaxation of MPQ. Later, we show that the linear programming relaxation of MPQ* does imply these inequalities.

The previous inequalities suggest the following more general class of inequalities that are based on the concept of jump-sets of arcs (see Dahl (1999) and Godinho, Gouveia and Magnanti (2007)). Let S_0, S_1,...,S_{k+1} be node-disjoint nonempty sets defining a partition of the node set V with S_0 = {1} and S_{k+1} = {i}. Consider the inequalities

$$\begin{aligned} f_{ij} \le (Q-1-k)x_{ij} + \sum_{p=0}^{k\text{-}1} \sum_{q=p+2}^{k+1} (q-p-1)x(S_p,S_q) \quad (i,j) \in A, k = 1,...,Q\text{-}2 \\ and\ (S_0,...,S_{k+1})\ of\ V\ with\ S_0 = \{1\}\ and\ S_{k+1} = \{i\}. \end{aligned} \tag{5.3a}$$

Note that when k = 1 and 2, we obtain the inequalities (5.1a) and (5.2a) described previously. Note that if a variable corresponds to an arc "jumping" over t intermediate sets S_j (i.e., (q - p – 1) = t), then its coefficient equals t. It is not difficult to check that the term

$$\sum_{p=0,...,k\text{-}1} \sum_{q=p+2,...,k+1} (q-p-1)x(S_p,S_q)$$

appearing in the inequalities satisfies the conditions previously given for the generic term $Exp(x,k)$. In the next section, we show how to generalize inequalities (5.3a) for arc sets and we will also discuss similar lower bounding inequalities

5.2 Generalizing constraints (5.3a) for Arc Sets

In this subsection we will analyze constraints of the form

$$f(S',S) \leq (Q-1-k)x(S',S) + ?? \quad S,S' \subset V \setminus \{1\}, S \cap S' = \phi \tag{5.4a}$$

that are stronger than the constraints obtained by adding (5.3a) for all the arcs in a cut [S',S]. For simplicity, we start with the simplest case, the following generalization of inequality (5.1a).

$$f(S',S) \leq (Q-2)x(S',S) + x(1,S') \quad S,S' \subset V \setminus \{1\}, S \cap S' = \phi. \tag{5.5a}$$

The validity of these constraints is easy to establish. The key item in constraints (5.5a) is the coefficient 1 of the righthand term for every pair of sets S and S'. These constraints are stronger than the ones obtained by adding |S|*|S'| corresponding constraints (5.1a) and this fact will prove to be crucial when, in the next subsection, we use "projection" techniques to generate new inequalities in the space of the variables x_{ij}.

It is not difficult to see how to generalize any of the constraints (5.3a) in a similar way to

$$\begin{gathered} f(S',S) \leq (Q-1-k)x(S',S) + \sum_{p=0}^{k\text{-}1}\sum_{q=p+2}^{k+1}(q-p-1)x(S_p,S_q) \quad S,S' \subset V \setminus \{1\}, S \cap S' = \phi, \\ k = 1,\ldots,Q-2 \textit{ and partitions } (S_0,\ldots,S_{k+1}) \textit{ of } V \textit{ with } S_0 = \{1\} \textit{ and } S_{k+1} = S'. \end{gathered} \tag{5.6a}$$

Again, we note that the coefficients of the variables within the righthand side summation are independent of S and S'. Thus, for a given k, S and S', the generalized inequality is stronger than the constraint obtained by adding |S|*|S'|| single node set constraints (5.3a).

In a similar way we write the following set of lower bounding inequalities:

$$\begin{gathered} f(S,S') \geq (k+1)x(S,S') - \sum_{p=0}^{k-1}\sum_{q=p+2}^{k+1}(q-p-1)x(S_p,S_q) \; S,S' \subset V \setminus \{1\}, S \cap S' = \phi, \\ k = 1,\ldots,Q-2 \textit{ and partitions } (S_0,\ldots,S_{k+1}) \textit{ of } V \textit{ with } S_0 = S' \textit{ and } S_{k+1} = \{1\}. \end{gathered} \tag{5.6b}$$

The following result (with a proof given in the Appendix) shows that a special case of (5.6a/b) is implied by the linear programming relaxation of MPQ. It is still open whether all inequalities (5.6a/b) are implied by the linear programming relaxation of MPQ.

Proposition 5.1 The inequalities (5.6a) (resp. (5.6b)) are implied by the linear programming relaxation of MPQ for all values Q if one of the following conditions are satisfied:

i) k = 1

ii) k = 2,…,Q-2 and $S \subseteq S_1 \cup S_2$ (resp. $S \subseteq S_{k-1} \cup S_k$).

To conclude this section, we show that if we make some assumptions on the partitions involved in the different inequalities, we can obtain inequalities that are stronger versions of inequalities that result from adding either several inequalities (5.6a) or (5.6b). Again, these inequalities might prove to be relevant when using projection techniques such as the ones described in Section 6 for obtaining inequalities defined in the space of the x_{ij} variables that are implied by the linear programming relaxation of MPQ. To motivate this set of constraints consider, the following three constraints (5.6a) for k = 4, 3 and 2, respectively, and for a specific family of pairwise disjoint node sets S1, S2, S3, S4 and S:

$$\begin{aligned} f(S4,S) &\le (Q-4)x(S4,S) + \sum_{p=0}^{2}\sum_{q=p+2}^{4}(q-p-1)\,x(A_p,A_q) \\ f(S3,S) &\le (Q-3)x(S3,S) + \sum_{p=0}^{1}\sum_{q=p+2}^{3}(q-p-1)\,x(B_p,B_q) \\ f(S2,S) &\le (Q-2)x(S2,S) + x(1,S2). \end{aligned}$$

In the first constraint, the node sets in the partition are $A_0 = \{1\}$, $A_1 = S1 \cup S$, $A_2 = S2$, $A_3 = S3$ and $A_4 = S4$. In the second constraint, the node sets in the partition are $B_0 = \{1\}$, $B_1 = S1 \cup S$, $B_2 = S2$ and $B_3 = S3$. Note that the sets S1, S2, S3 and S4 appear in increasing order in these three partitions. In such a case, it is possible to derive the inequality

$$\begin{aligned} f(S4,S) + f(S3,S) + f(S2,S) \le{}& (Q-4)x(S4,S) + (Q-3)x(S3,S) + (Q-2)x(S2,S) \\ &+ \sum_{p=0}^{2}\sum_{q=p+2}^{4}(q-p-1)\; x(A_p,A_q). \end{aligned}$$

This inequality is stronger than adding the previous three inequalities since it does not include the rightmost terms of the second and third inequalities. This inequality is easily generalized yielding the following inequalities:

$$\begin{aligned} &\sum_{u=2}^{k+1} f(S_u,S) \le \sum_{u=2}^{k+1}(Q-u)x(S_u,S) + \sum_{p=0}^{k\text{-}1}\sum_{q=p+2}^{k+1}(q-p-1)x(A_p,A_q) \quad k = 2,\dots,Q-2 \\ &\quad \textit{and partitions } (S_1,\dots,S_{k+1},S) \textit{ of } V\setminus\{1\} \textit{ such that} \\ &\quad A_0 = \{1\}, A_1 = S_1 \cup S \textit{ and } A_u = S_u \ (u = 2,\dots,k+1). \end{aligned} \tag{5.7a}$$

In a similar manner we can write lower bounding inequalities as follows:

$$\sum_{u=2}^{k+1} f(S_u, S) \geq \sum_{u=2}^{k+1} u\, x(S_u, S) \quad - \sum_{p=0}^{k\text{-}1} \sum_{q=p+2}^{k+1} (q-p-1)x(A_p, A_q) \quad k = 1,...,Q-2$$

$$\textit{and partitions } (S_1,...,S_{k+1},S) \textit{ of } V \textit{ such that} \qquad (5.7b)$$

$$A_0 = S_{k+1}, A_u = S_{k-u} \ (u = 1,...,k-1),\ A_k = S_1 \cup S, \textit{and } A_{k+1} = \{1\}.$$

The following result (with a proof given in the Appendix) shows that these inequalities are implied by the linear programming relaxation of MPQ.

Proposition 5.2 The inequalities (5.7a/b) are implied by the linear programming relaxation of MPQ.

5.3 Some Inequalities Implied by MPQ* in the x_{ij} and f_{ij} Space

Some (and perhaps) all the inequalities derived in the previous subsections are implied by the linear programming relaxation of the stronger model MPQ. Finding inequalities that in the space of the variables x_{ij} and f_{ij} that are implied by the linear programming relaxation of MPQ* but are not implied by the linear programming relaxation of MPQ is also of interest but does not seem to be an easy task. Here we produce one such set of inequalities. In Subsection 5.1 we have mentioned the following slightly stronger form of inequalities (5.2a):

$$f_{ij} \leq (Q-3)x_{ij} + 2x_{1i} + x(1,P) + x(V \setminus (P \cup \{1,j\}),i) \quad (i,j) \in A,\ i,j \neq 1, P \subseteq V \setminus \{1,i,j\}. \qquad (5.8a)$$

As noted before, these inequalities differ from (5.2a) by considering more slightly restricted index sets in the last term and are not implied by the linear programming relaxation of MPQ. We can also write a similar and slightly stronger form of the corresponding lower bounding inequalities (5.2b) as follows:

$$f_{ij} \geq 3x_{ij} - 2x_{j1} - x(P,1) - x(j, V \setminus (P \cup \{1,i\})) \quad (i,j) \in A,\ i,j \neq 1, P \subseteq V \setminus \{1,i,j\}. \qquad (5.8b)$$

Proposition 5.3: The inequalities (5.8a/b) are implied by the linear programming relaxation of the MPQ* model.

It does not seem easy to generalize (5.8a) in the same way that we generalized constraints (5.2a) for sets of nodes (see (5.6a) for k = 2). It is also not clear how to use the constraints (3.1) to obtain stronger versions of the upper and lower bounding constraints for arcs that are more than 3 arcs away from the depot.

6. Some Inequalities implied by MPQ in the x_{ij} space

In this section we describe several inequalities in the space of the variables x_{ij} that are implied by the linear programming relaxation of the MPQ model. Although à priori we do not know the structure of such inequalities, fortunately, we can use the generalized flow bounding constraints of the single commodity flow model derived in the previous section to suggest these inequalities. These new inequalities will be obtained either in a trivial way (see Subsection 6.1) or by using projection techniques similar to the ones described in Gouveia (1995) and Letchford and Salazar-Gonzalez (2006) which will be described in Subsection 6.2. Note that the way these inequalities are obtained shows that they are also implied by the linear programming relaxation of the SCF model augmented with (5.6a/b) and/or (5.7a/b) in the space of the x_{ij} variables. We will also show that with exception to a few cases, we can obtain stronger inequalities by finding direct derivations from MPQ. We note, however, that the new flow bounding inequalities together with the known projection techniques have suggested the nature of some of the inequalities implied by the linear programming relaxation of MPQ.

6.1 Conditional Double-Jump Inequalities

The first set of inequalities is quite simple. Adding the upper bound constraint (5.6a) and the lower bound constraint (5.6b) for k = 1 and any arc cut set [S',S] and canceling equal terms, we obtain the constraints

$$0 \le (Q-4)x(S',S) + x(1,S') + x(S,1) \quad S,S' \subseteq V\setminus\{1\}, S\cap S' = \phi \tag{6.1}$$

which apparently are uninteresting since for Q > 3 they are redundant. However, when Q = 3 we obtain the intuitive inequality $x(S',S) \le x(1,S') + x(S,1)$. Note that if we set S' = {2} and S = {6}, the solution shown in Figure 5.1 violates the resulting inequality (6.1). We can obtain more complicated inequalities by combining other general lower and upper bounding inequalities from the entire class (5.6a/b). However, we believe that, in general, those inequalities are easily dominated by other inequalities implied by the linear programming relaxation of MPQ. As an example, if we combine (5.6a) for k = 2 with (5.6b) for k = 1 to eliminate the term $f(S',S)$, we obtain, after rearranging,

$$\begin{gathered} 0 \le (Q-5)x(S',S) + 2x(1,S') + x(1,P) + x(V\setminus(P\cup\{1\}),S') + x(S,1) \\ S,S' \subseteq V\setminus\{1\}, S\cap S' = \{\}, P \subseteq V\setminus\{1\cup S'\}. \end{gathered} \tag{6.2}$$

When Q = 4, the inequality (6.2) becomes

$$\begin{gathered} x(S',S) \le 2x(1,S') + x(1,P) + x(V\setminus(P\cup\{1\}),S') + x(S,1) \\ S,S' \subseteq V\setminus\{1\}, S\cap S' = \{\}, P \subseteq V\setminus\{1\cup S'\}. \end{gathered} \tag{6.3}$$

However, it is easy to see how to strengthen this inequality by replacing the coefficient of "2" of the term $x(1,S')$ with a "1". These resulting inequalities belong to a more general class of inequalities that are implied by the linear programming of MPQ but, besides a few special cases (for example, the inequalities

$x(S',S) \leq x(1,S') + x(S,1)$ for Q = 3), do not appear to be implied by the linear programming relaxation of SCF augmented by inequalities (5.6a/b) and/or (5.7a/b).

These inequalities are similar to the double-jump inequalities introduced in Godinho, Gouveia and Magnanti (2007). To motivate the new inequalities, assume that an arc (i,j) (with i,j ≠ 1) is in the solution in one of the routes. Then, for any position p ($2 \leq p < Q$), either the path from node 1 to i has length less or equal than p-1, or the path from node j to node 1 has length less or equal than Q-p. This observation suggests the following conditional version of a double jump inequality. Let $S_0, S_1,...,S_p$ define a partition of the node set V with $S_0 = \{1\}$, $S_p = \{i\}$. Let $[S_j, S_{j+1}] = \{(u,v) \in A: u \in S_i, v \in S_j\}$ and define $J1 = J(S_0, S_1,..., S_p) = \cup_{[i+1<j]} [S_i, S_j]$. Let $S'_0, S'_1,..., S'_{Q-p+1}$ define another partition of the node set V with $S'_0 = \{j\}$ and $S'_{Q-p+1} = \{1\}$ and define $J2 = J(S'_0, S'_1,..., S'_{Q-p+1}) = \cup_{[i+1<j]} [S_i, S_j]$. For a given arc (i,j) and position p ($2 \leq p < Q$), we consider a *conditional double jump set of arcs* defined as $CDJ = J(S_0, S_1,..., S_p) \cup J(S'_0, S'_1, \ldots, S'_{Q-p+1})$. We refer to such a set CDJ as a ((i,j),Q,p)-*conditional double jump* and we let $CDJ_{((i,j),Q,p)}$ denote the set of all ((i,j),Q,p)-conditional double jumps. To explain the constraints to be presented next, assume that arc (i,j) is in the solution and consider a fixed position p. Then, any vehicle path from the depot to node i that contains an arc from every set $[S_j, S_{j+1}]$ taken from $J(S_0, S_1,..., S_p)$ must contain at least p arcs and any vehicle path from node j to the depot that contains an arc from every set $[S_j, S_{j+1}]$ taken from $J(S'_0, S'_1, \ldots, S'_{Q-p+1})$ takes at least Q-p+1 arcs. These two paths together with the arc (i,j) cannot constitute a feasible route. Thus, either the first path contains at most p-1 arcs (and at least one arc from a "jump set" $[S_i, S_j]$ with $i+1 < j$ taken from $J(S_0, S_1,..., S_p)$) or the second path contains at most Q-p arcs (and at least one arc from a "jump set" $[S_i, S_j]$ with $i+1 < j$ taken from $J(S'_0, S_1,..., S_{Q-p+1})$). These arguments show that the following inequalities are valid,

$$x(CDJ) \geq x_{ij} \qquad \text{for all } (i,j) \in A,\ i,j \neq 1 \text{ and } CDJ \in CDJ_{((i,j),Q,,p).} \qquad (6.4)$$

Notice that when Q = 3, this inequality becomes the constraint $x_{ij} \leq x_{1i} + x_{j1}$ and when Q = 4 and h = 3 we obtain the stronger version of the inequality (6.3). A simple generalization of the previous argument shows that constraints (6.4) can also be generalized for arc sets. To define these more general constraints we need to generalize a little bit, the concept of conditional double jump. For a given arc set (A,B) with $A \cap B = \phi$ and $A,B \subseteq V \backslash \{1\}$, and position h ($2 \leq h < Q$), we consider, now, a *conditional double jump set of arcs* defined as $CDJ = J(S_0, S_1,..., S_p) \cup J(S'_0, S'_1, \ldots, S'_{Q-p+1})$ as before but, this time, setting $S_p = A$ and $S'_0 = B$. We refer to such a CDJ set as a ((A,B),Q,h)-*conditional double jump* and we let $CDJ_{((A,B),Q,p)}$ denote the set of all ((A,B),Q,p)-conditional double jump. The inequalities are:

$x(CDJ) \geq x(A,B)$ for all A,B $\subseteq$ V\{1}, A $\cap$ B = ϕ and CDJ $\in$ $CDJ_{((A,B),Q,p).}$ (6.4')

Note that constraints (6.4') contain constraints (6.4) as special cases. The next result, proved in the Appendix, states that some of these inequalities are implied by the linear programming relaxation of MPQ. It is still open whether all of the inequalities (6.4') are implied by the linear programming relaxation of MPQ.

Proposition 6.1 The inequalities (6.4') are implied by the linear programming relaxation of MPQ for all Q if one of the following conditions hold:

iii) $p \leq 3$

iv) $p > 4$, $B \subseteq S_1 \cup S_2$ and $A \subseteq S'_{Q-p-1} \cup S'_{Q-p}$.

We note from the conditions in the statement of the proposition that the result is valid when Q = 3 and 4 independent of the value of p.

6.2 Flow Based Projected Inequalities

We can obtain a different class of inequalities by using projection techniques applied to the SCF model augmented with the flow bounding inequalities derived in Section 5. We start by reproducing the procedure used by Gouveia (1995) (see also Letchford and Salazar-Gonzalez (2006)) for generating the multistar inequalities. By adding the flow conservation constraints for nodes j in a set S $\subseteq$ V\{1} (for simplicity, let S' denote the set V\(S$\cup$1)) we obtain:

$$f(1,S) + f(S',S) = |S| + f(S,S') \qquad \text{for all } S \subseteq V\backslash\{1\}. \qquad (6.5)$$

Using the flow upper bound constraints $f_{1j} \leq Qx_{1j}$ and $f_{ij} \leq (Q-1)x_{ij}$ on the lefthand side, the flow lower bound constraints $f_{ij} \geq x_{ij}$ on the righthand side, and the assignment constraints, we obtain the multistar constraints

$$Qx(S) + x(\delta(S)) \leq |S|(Q-1). \qquad \text{for all } S \subseteq V\backslash\{1\}. \qquad (6.6)$$

In this expression $\delta(S)$ denotes the set of arcs with only one endpoint in S, excluding arcs directed into or out of the depot node 1. As noted by Letchford and Salazar-Gonzalez (2006), we can use the same procedure to obtain projected inequalities for several versions of the CVRP problem, including variations with pick-up and delivery and variations with distance constraints.

In our context we could obtain many different projected inequalities for the CVRP by using the same procedure and bounding the flow sums in equation (6.5) with alternate expressions (5.6a/b)) (note that since the first summation of expression (6.5) is always bounded by the same expression (Q $x(1,S)$), we will not consider options for it in this discussion). For the moment, we will consider only the following four upper bound and lower bound constraints for arcs not entering or leaving node 1. Constraints (2) and (4) correspond to the inequalities (5.5a/b) for k = 1.

$$
\begin{array}{lll}
(1) & f(A,B) \leq (Q-1)x(A,B) & A,\ B \subseteq V\setminus\{1,\}, A\cap B=\phi \\
(2) & f(A,B) \leq (Q-2)x(A,B) + x(1,A) & A,\ B \subseteq V\setminus\{1,\}, A\cap B=\phi \\
(3) & f(A,B) \geq x(A,B) & A,\ B \subseteq V\setminus\{1,\}, A\cap B=\phi \\
(4) & f(A,B) \geq 2x(A,B) - x(B,1) & A,\ B \subseteq V\setminus\{1,\}, A\cap B=\phi.
\end{array}
$$

However, a straightforward and naïve use of this procedure will lead to new, but uninteresting inequalities in the sense that (see Godinho et al. (2007)) all the inequalities obtained by using the same upper bound expression (1) and (2) for all the arcs in the cuts [S',S] and the same lower bound expression (3) and (4) for the arcs in the cut [S,S'] are implied by the multistar constraints (6.6). Fortunately, we can generalize this procedure by partitioning the cut sets into two subsets and using different upper (or lower) bounds for each partition. Suppose we partition the set S' into two sets S1 and S2 and for the arcs in the cut [S1,S] we bound the flow by using the original upper bound inequality (1) and for the arcs in the cut [S2,S] we bound the flow using the inequalities (2). We can perform a similar manipulation with the summation on the righthand side by partitioning the cut [S,S'] into two cuts [S,S1] and [S,S2]. Table 6.1 shows how to obtain, after some manipulation using the assignment constraints, symmetric inequalities and non-symmetric inequalities using this procedure.

Option for Bounding $f(S1,S)+f(S2,S)$	Option for Bounding $f(S,S1)+f(S,S2)$	Resulting Inequality	
(1) + (2)	(3) + (4)	$Qx(S)+2x(S2)+x(\delta(S))+x(\delta(S2)) + x(S,S2)+x(S2,S) \leq \lvert S\rvert(Q-1)+2\lvert S2\rvert$	(6.7)
(1) + (2)	(4) + (3)	$Qx(S)+x(S1)+x(S2)+2x(\delta(S)) + + 2x(S1,S2)) \leq \lvert S\rvert(Q-1)+\lvert S1\rvert+\lvert S2\rvert$	(6.8)

Table 6.1. Inequalities generated from bounding aggregated flow constraints.

We note that when either S1 or S2 is empty, we obtain dominated inequalities (since these inequalities are obtained by using the same expression for all the arcs in the cuts [S',S] and [S,S']). Note also that when S2 is empty, the inequalities (6.7) become the multistar constraints (6.6).

To see how effective the constraints (6.7) and (6.8) might be, we tested them on small problem instances with n = 5, 6 and 7, Q = 3, 4 and 5 and almost all choices of |S1|, |S2| and |S| (we conducted the test by solving the linear programming relaxation of the SCF model with an objective function equal to the left hand side of the constraint (6.7) or (6.8)) For these small networks inequalities (6.7) were implied by the multistar constraints (6.6) for Q = 3. In addition for Q = 3, we can formally show that the inequality (6.7) is redundant since it is implied by the multistar constraint (6.6) for the set (S ∪ S2). However, for Q > 3, we can find solutions that are feasible for the linear programming relaxation of SCF but are violated by (6.7). In contrast, for many of these small instances, the asymmetric instances (6.8) were not dominated. As an example, the solution depicted in Figure 5.1 violates the asymmetric constraint (6.8) for S={2,6}, S_1={5} and S_2= {3,4}.

We can obtain a more general set of constraints (that we will not describe here – see Godinho et al. (2007)) if we generalize the previous procedure by partitioning the cut [S',S] into two cuts [S1,S] and [S2,S] and the cut [S,S'] into two cuts [S,S3] and [S,S4] (S3 does not necessarily equal S1 or S2). The inequalities (6.7) and (6.8) are particular cases of these more general inequalities, obtained when S1 = S3 and S2 = S4 or S1 = S4 and S2 = S3.

We have to this point partitioned the set S' in the expression (6.5). Instead we can partition the set S into two sets S1 and S2. We note (but do not include the proof here – again, see Godinho et al (2007)) that i) inequalities obtained by partitioning the cut [S',S] into two cuts [S',S1] and [S',S2] and partitioning the cut [S,S'] into two cuts [S3,S'] and [S4,S'] and ii) inequalities obtained by partitioning either the UB term or the LB term (but not both), also lead only to inequalities that are implied by the multistar constraints.

To this point, in our effort to identify inequalities implied by the model MPQ we have partitioned the set S' in the flow expression (6.5) into two sets and used the simpl3 upper and lower bounds (1) - (4). To uncover other, more complicated, inequalities we might use partitions with more sets and use the more complicated jump bounds (5.7a/b). There are many such possibilities. As an illustration consider inequalities that arise when we partition the set S' in the cuts [S',S] and [S,S'] into three subsets S1, S2 and S3. Once we set the (arbitrary) order S1, S2, S3 for the upper bounding, we have six different options for ordering the sets in the lower bounding. We won't consider all possibilities, but will develop three sets of inequalities, one generalizing each of the inequalities (6.7) and (6.8) and one with a new form.

For the bounding we will use the simple upper and lower bounds (1) and (3) for some of the flows and the following four sets of inequalities, one a jump upper bound (5.7a) and three alternate jump lower bounds (5.7b):

$$(5)\quad f(S2,S)+f(S3,S)\le(Q-2)x(S2,S)+(Q-3)x(S3,S)+\sum_{p=0}^{2}\sum_{q=p+2}^{3}(q-p-1)x(A_p,A_q)$$
$$with\ A_0=\{1\},A_1=S_1\cup S,A_2=S_2\ and\ A_3=S_3$$

$$(6)\quad f(S,S2)+f(S,S3)\geq 2x(S,S2)+3x(S,S3)+\sum_{p=0}^{2}\sum_{q=p+2}^{3}(q-p-1)x(A_p,A_q)$$
$$\text{with } A_0=S_3, A_1=S_2, A_2=S_1\cup S \text{ and } A_3=\{1\}$$

$$(7)\quad f(S,S1)+f(S,S3)\geq 2x(S,S1)+3x(S,S3)+\sum_{p=0}^{2}\sum_{q=p+2}^{3}(q-p-1)x(A_p,A_q)$$
$$\text{with } A_0=S_3, A_1=S_1, A_2=S_2\cup S \text{ and } A_3=\{1\}$$

$$(8)\quad f(S,S2)+f(S,S1)\geq 2x(S,S2)+3x(S,S1)+\sum_{p=0}^{2}\sum_{q=p+2}^{3}(q-p-1)x(A_p,A_q)$$
$$\text{with } A_0=S_1, A_1=S_2, A_2=S_3\cup S \text{ and } A_3=\{1\}.$$

Note that the coefficients q – p – 1 in these inequalities will be at most two.

By using the inequalities (1) with A = S1 and B = S and (5) for the upper bounds, and (3) with A=S and B = S1 and (6) for the lower bounds, we obtain the following generalization of (6.7)

$$\begin{aligned}&Qx(S)+2x(S2)+4x(S3)+x(\delta(S))+x(\delta(S2))+x(\delta(S3))+x(S2,S1)+x(S1,S2)+\\&+x(S,S2)+x(S2,S)+x(S,S3)+x(S3,S)\leq|S|(Q-1)+2|S2|+4|S3|.\end{aligned}\qquad(6.9)$$

and by using the inequalities (1) with A = S1 and B = S and (5) for the upper bounds, and (3) with A = S and B = S2 and (7) for the lower bounds, we obtain the following generalization of (6.8):

$$\begin{aligned}&Qx(S)+x(S1)+x(S2)+4x(S3)+2x(S1,S2))+2x(\delta(S))+2x(\delta(S3)\leq\\&\leq|S|(Q-1)+|S1|+|S2|+4|S3|.\end{aligned}\qquad(6.10)$$

Note that when S3 is empty, the previous inequalities reduce to (6.7) and (6.8). Finally, by using the inequalities (1) with A = S1 and B = S and (5) for the upper bounds, and (3) with A = S and B = S3 and (8) for the lower bounds, we obtain the following inequality:

$$\begin{aligned}&Qx(S)+2[x(S1)+x(S2)+x(S3)]+2x(\delta(S))+x(S,S1\cup S2)+x(S2\cup S3,S)+2x(S1,S3)\\&+3[x(S1,S2)+x(S2,S3)]+x(S3,S2)+x(S2,S1)\leq|S|(Q-1)+2(|S1|+|S2|+|S3|).\end{aligned}\qquad(6.11)$$

These inequalities demonstrate something we have mentioned in the introduction of this section, namely we can tighten some inequalities obtained by manipulating inequalities from SCF augmented with the constraints (5.6a/b) or (5.7a/b) by using direct derivations from the model MPQ. In fact, for the specific case of Q = 3, we can tighten inequalities (6.7) and (6.8). As we have noted before the inequality (6.7) is dominated by a multistar constraint and inequality (6.8) is dominated by the following inequality

$$3x(S)+x(S1)+x(S2)+2x(\delta(S))+2x(S1,S2)+x(S2,S1)\leq 2|S|+|S1|+|S2|\qquad(6.12)$$

which is implied by the linear programming relaxation of MPQ. Something similar happens with inequalities (6.9), (6.10) and (6.11) and Q = 4. For instance, the following inequality is implied by the linear programming relaxation of MPQ:

$$\begin{aligned}4x(S)+2[x(S1)+x(S2)+x(S3)]+3x(\delta(S))+\ 3[x(S1,S2)+x(S2,S3)+x(S1,S3)]\\ +2[x(S2,S1)+x(S3,S2)+x(S3,S1)]\le 3\,|\,S\,|+2(|\,S1\,|+|\,S2\,|+|\,S3\,|),\end{aligned} \tag{6.13}$$

and it implies the inequality (6.11) when Q = 4. We refer to Godinho et al. (2007) for a proof that these two inequalities, (6.12) and (6.13) are implied by the linear programming relaxation of MPQ.

All of these inequalities are implied by the linear programming relaxation of MPQ. However, they can be used to generate inequalities that are not implied by the linear programming relaxation of MPQ. As an illustration, recall the following well-known fact. If, for a given set S, we divide the corresponding multistar constraint by Q and apply integer rounding, we obtain the generalized subtour elimination constraint (see Section 2) for the set S. Applying the same procedure to constraints (6.9), (6.10) and (6.11) for a given set S leads to the same inequality. However, if for a given set S, we divide inequality (6.10) by 2 and use integer rounding we obtain the following inequality

$$\begin{aligned}\lfloor Q/2\rfloor x(S)+2x(S3)+x(S1,S2))+x(\delta(S))+x(\delta(S3)\le\\ \le\lfloor(|\,S\,|\,(Q-1)+|\,S1\,|+|\,S2\,|+4\,|\,S3\,|)/2\rfloor.\end{aligned} \tag{6.14}$$

A few experiments with small instances for Q = 3, 4 and 5 have indicated that the inequalities (6.14) are not implied by the linear programming relaxation of MPQ. In Figure 6.1 we indicate one such solution for a situation when Q = 3 and S3 is empty. It is easy to see that the solution shown in the figure violates constraint (6.14) for $S=\{7\}$, $S_1=\{5,6\}$ and $S_2=\{2,3,4\}$. Note also that this solution does not violate any of the generalized subtour elimination constraints.

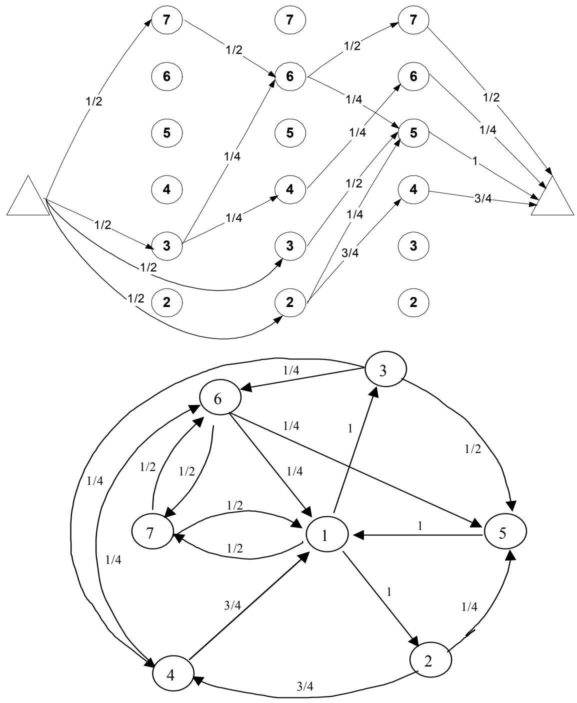

Figure 6.1 – A feasible solution for the MPQ_L model that violates inequality (6.14). The topmost figure shows the values of the z variables and the bottommost the values of the x variables.

In a similar manner we can generate a rounded inequality from (6.11) leading to

$$\lfloor Q/2 \rfloor x(S) + x(S1) + x(S2) + x(S3) + x(\delta(S)) + x(S1,S3) + x(S1,S2) + x(S2,S3) \\ \leq \lfloor (|S|(Q-1) + 2(|S1| + |S2| + |S3|))/2 \rfloor \tag{6.15}$$

We can also obtain rounded inequalities (6.12) for Q = 3 and (6.13) respectively for Q = 4. However, these inequalities will be equivalent to (6.14) with Q = 3 and to (6.15) with Q = 4.

7. Computational Results

In this section we empirically evaluate the quality of the lower bounds given by the two time-dependent models presented in the previous sections. For comparing the models, we use data for complete graphs with 40 and 80 nodes plus the depot. We have considered three test sets. For the first two sets, the nodes have been randomly selected in a square grid with Euclidean arc lengths. These tests are divided according to the location of the depot. The designation TEM refers to Euclidean cost instances with the depot located in the middle of the grid and the designation TEC refers to Euclidean cost instances with the depot located in the corner. We have also considered two sets of random costs instances, one set with symmetric edge costs (denoted by TRS) and another set with asymmetric cost instances (denoted by TRA). The costs of the edges of the TRS instances were generated from an uniform distribution in the interval [1,100] while the costs of the arcs of the TRA instances were generated from an uniform distribution in the same interval. Each group includes 5 complete graph instances.

The models that we are comparing are the models SCF, MPQ, MPQ* and SCF+ (which is SCF augmented with inequalities (5.1a/b) of Section 5). Table 7.1 presents the average linear programming gaps (computed as [(Optimal Value – $v(P_L)$)/Optimal value]*100 for each group. We solved the linear programming relaxations using the CPLEX 9.0 software package on a Pentium IV, 3.4GHz computer with 1Gb of RAM, and obtained the optimal solutions by running the model MPQ* with the branch-and-cut algorithm of CPLEX. Table 7.2 presents the CPU times for solving the linear programming relaxations of each model and for finding the integer solution.

	n	Q	SCF	SCF+	MPQ	MPQ*
TEM	41	3	3,22	3,22	3,22	1,09
		5	8,59	7,66	4,94	1,95
	81	3	2,02	2,02	2,02	0,51
		5	5,42	3,61	2,39	0,94
TEC	41	3	2,96	2,96	2,96	0,65
		5	5,40	3,79	2,87	1,10
	81	3	1,48	1,48	1,48	0,42
		5	3,90	2,84	2,22	0,82
TRS	41	3	2,45	2,45	2,45	1,32
		5	5,79	4,30	4,16	0,68
	81	3	2,62	2,62	2,62	2,31
		5	4,29	2,36	2,35	0,19
TRA	41	3	5,57	3,30	3,03	2,74
		5	7,14	5,37	2,56	2,08
	81	3	3,30	1,32	1,26	1,38
		5	5,88	3,82	1,32	1,31

Table 7.1 – Average gaps of the linear programming relaxation bounds of several models.

	N	Q	SCF	SCF+	MPQ	MPQ*	
TEM	41	3	1,17	4,71	0,05	0,24	6,86
		5	2,22	4,41	0,19	1,78	2755,47
	81	3	12,14	237,59	0,31	2,35	190,40
		5	22,03	118,57	2,34	24,35	76504,92
TEC	41	3	1,29	6,13	0,05	0,84	3,28
		5	1,25	5,64	0,25	2,32	684,80
	81	3	14,39	151,25	0,31	2,47	598,81
		5	22,78	140,06	2,10	31,69	97879,57
TRS	41	3	1,74	6,32	0,04	0,27	0,6
		5	1,68	4,23	0,35	2,79	13,2
	81	3	15,01	138,22	0,28	2,67	125,1
		5	21,54	141,94	4,21	41,57	1491,0
TRA	41	3	1,70	6,07	0,05	0,24	0,6
		5	1,55	3,18	0,31	1,78	3,4
	81	3	17,48	103,74	0,32	2,27	16,1
		5	21,94	70,20	5,05	1,31	34,8

Table 7.2 – Average CPU times in seconds of the linear programming relaxation bounds of several models. The rightmost column specifies the CPU times for obtaining the optimal integer solutions with model MPQ*.

The single commodity models contain $2*n^2$ variables and the time-dependent models contains $Q*n^2$ variables. The MPQ model contains Q*n constraints and the MPQ* contains $Q*(n + n^2)$ constraints. Therefore, the largest model being solved (MPQ*) contains more than 32,805 variables and 32,400 constraints.

The results indicate that the linear programming relaxation of the model MPQ produces reasonable improvements over the linear programming of SCF for the Q = 5 instances. For Q = 3 the reported gaps for all the symmetric instances (instances TEM, TEC and TRS) are the same for the two models. However, we obtain reasonable improvements for the TRA instances. The discrepancy between asymmetric and symmetric results reported for Q = 3 conform to our analysis (see Section 6) of the inequalities implied from the linear programming relaxation of MPQ.

The results also indicate that for the instances with Q = 5 (where the linear programming relaxation of MPQ improves upon the linear programming relaxation of SCF), the gap reduction from model SCF obtained by the linear programming relaxation of the model SCF+ is about 60 percent of the gap reduction obtained by the linear programming relaxation of the model MPQ (with a range of 25 to 99 percent). For Q = 3 and the asymmetric instances, it appears that the simple flow bounding inequalities (5.1a/b) are very effectives since the gaps produced by SCF+ are quite close to the ones obtained by MPQ.

The computations also show a difference between the gaps provided by the two hop indexed models MPQ and MPQ* for the symmetric and asymmetric instances (recall that MPQ* is obtained by adding inequalities (3.1) to the MPQ model). For symmetric instances the addition of these inequalities leads to a reduction of about 50 per cent from the linear programming relaxation gaps of the original MPQ model. The model MPQ* seems to be a good option for solving small and medium sized symmetric instances (up to 81 nodes) whenever Q is reasonably small. In these instances, the model is very attractive to use within an ILP package because it is small and it provides reasonable lower bounding values. To obtain the optimal integer solutions, the average CPU times do not show the high variability of the individual problem instances as compared with the variability from one group of problem instances to another. For instance, for the EM instances with n = 81 and Q = 3, one instance is solved within 100 seconds while another is solved after 40,000 seconds. We might note that the computational results we have obtained are not competitive with other, more intricate methods (as for instance, the methods described in Lysgaard, Letchford and Eglese (2004) and Fukasawa et al. (2006)). However, it is worth having a model that essentially anyone can implement and that can solve instances within certain parameters (i.e., small values of Q)..

Perhaps, one way of explaining why adding constraints (3.1) to MPQ is not that effective for the asymmetric cases, is to see why it is effective for the symmetric cases. Note that for a symmetric instance, an attractive “edge” {i,j} (one with a small cost) is very likely used in the two directions, allowing both of the variables z_{ij}^{t} and z_{ij}^{t+1} (for a certain t) to be simultaneously positive. Inequalities (3.1) prohibit these situations. Note that these situations are less likely to occur in asymmetric instances due to the cost structure.

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Appendix

In this Appendix we prove some of the Propositions stated in the text. For any pair of node sets A and B and for any h, h=2,3,..,Q we let $z^h(A,B)$ denote $\sum_{i\in A,j\in B} z_{ij}^h$ and let $z^h(A)$ denote the case when A = B. We start by proving Proposition 6.1 since it is simpler than the proof of Propositions 5.1 and 5.2.

Proposition 6.1 The inequalities (6.4') are implied by the linear programming relaxation of MPQ for all Q if one of the following holds:

i) $p \leq 3$

ii) $p > 4$, $B \subseteq S_1 \cup S_2$ and $A \subseteq S'_{Q-p-1} \cup S'_{Q-p}$.

Proof: Let $J(S_0, S_1, ..., S_{p-1}, S_p)$ and $J(S'_0, S'_1, ..., S'_{Q-p}, S'_{Q-p+1})$ be jump sets as specified in Section 6. First, note that $x(A,B) = \sum_{h=2,...,Q} z^h(A,B) = \sum_{h=2,...,p} z^h(A,B) + \sum_{h=p+1,...,Q} z^h(A,B)$ for any p =2,…,Q. Therefore, we can establish the desired inequality $x(CDJ) \geq x(A,B)$ if we can show that

$$\sum_{h=2,...,p} z^h(A,B) \leq \sum_{h=1,...,Q} \sum_{q=0,...,p-2} \sum_{r=q+2,...,p} z^h(S_q, S_r) = \sum_{h=1,...,Q} z^h(J(S_0, S_1, ..., S_{p-1}, S_p))$$

$$\sum_{h=p+1,...,Q} z^h(A,B) \leq \sum_{h=2,...,Q+1} \sum_{q=0,...,Q-p-1} \sum_{r=q+2,...,Q-p+1} z^h(S'_q, S'_r) = \sum_{h=2,...,Q+1} z^h(J(S'_0, S'_1, ..., S'_{Q-p}, S'_{Q-p+1}))$$

We actually will establish a stronger inequality with the index h = 2,…,Q replaced by h = 1,…,p-1 in the first inequality and h = 2,…,Q replaced by h=p+2,..,Q+1 in the second inequality.

In the previous expression we assume for simplification that p = 3,…,Q-1. When p = 2, the first inequality corresponds to the easily established inequality $z^2(A,B) \leq x(1,A)$ while when p = Q-1, the second inequality corresponds to the easily established inequality $z^Q(A,B) \leq x(B,1)$.

Next, we will prove the first inequality (the proof for other inequality is similar).

In the following, let S_{p+1} = B and recall that S_p = A and $S_0 = \{1\}$. The fact that the sets S_r (r = 1,…,p) are disjoint, the fact that $B \cap S_r = \phi$ for all r=3,…,p-1, and the flow equations for the z_{ij}^h variables in the MPQ formulation imply the following inequalities

$$\begin{aligned}\sum_{r=q+1,...,j+1} z^q(S_j, S_r) &= z^q(S_j, \bigcup_{r=q+1}^{r=j+1} S_r) \leq z^{q-1}(\bigcup_{r=1}^{r=p} S_r, S_j) = \\ &= z^{q-1}(\bigcup_{r=1}^{r=j-2} S_r, S_j) + z^{q-1}(\bigcup_{r=j-1}^{r=p} S_r, S_j)\end{aligned} \quad \text{for all (j,q) such that } (p \geq j \geq q \geq 3) \qquad \text{(A.1)}$$

The inequality is valid since the righthand side is the total flow into the set S_j while the left hand side is the partial flow out of this set. The first term after the final inequality contains valid jump terms and the final term contains invalid terms for the jump inequalities.

We write next an analoge of inequality (A.1) for q = 2. The validity of the inequality follows from the fact that the sets S_r (r = 1,…,p) are disjoint and from the flow equations for the z_{ij}^h variables in the MPQ formulation

$$\sum_{r=3,\dots,j+1} z^2(S_j,S_r) = z^2(S_j,S_3\cup\dots\cup S_{j+1}) \le z^1(1,S_j) \qquad \text{for all j such that } (p \ge j \ge 2) \qquad \text{(A.2)}$$

Note that for any fixed *u* and *t* with p ≥ u ≥ t-1 ≥ q-1, the term $z^{q-1}(S_u,S_t)$ appears in the righthand side of the inequality (A.1) for the pair (t,q) and appears in lefthand side of inequalities A.1 or A.2, according to the value of q. If q-1=2, it appears in the lefthand side of A.2 for the pair (u,q-1); if q-1 ≥ 3, it appears on the lefthand side of A2 for the pair (u,q-1). Note also that when j = p and S_p = A, the lefthand side of the inequality (A.1) becomes $z^q(A,S_{q+1}\cup S_{q+2}\cup\dots\cup S_{p-1}\cup A)+z^q(S_j,B)$ and the inequality is valid only if $B\cap S_r=\phi$ for all r=q+1,…,p-1. Otherwise, the first equality in the inequality (A.1) will not be valid since it would contain overlapping terms $z^q(S_p,S_r)$ and $z^q(S_p,B)$ for some r = q+1,…,p-1 and we would need to use a coefficient of 2 on the righthand side of the inequality.

By adding inequalities (A.1) for all (j,q) such that p ≥ j ≥ q ≥ 3 and inequalities (A.2) for all (j,2) withp ≥ j ≥ 2 and by canceling equal terms we obtain the desired inequality

To prove Proposition 5.1 we, first, develop two lemmas.

Lemma 1: The model MPQ_L implies the inequalities

$$f(A,B) \le (Q-1-k)x(A,B) + \sum_{h=2,\dots,k+1}(k-h+2)z^h(A,B)$$

$$\textit{for all } A,B\subseteq V-\{1\}, A\cap B=\emptyset,\ k = 1,\dots,Q-2 \text{ and}$$

$$f(A,B) \ge (k+1)\, x(A,B) - \sum_{h=Q-k+1,\dots,Q}(k+h-Q)z^h(A,B)$$

$$\textit{for all } A,B\subseteq V-\{1\}, A\cap B=\emptyset,\ k = 1,\dots,Q-2$$

Proof: We will provide the proof for the upper bound constraints. The proof for the lower bound constraints is similar and is omitted.

Using (4.2) and the linking constraints of MPQ_L, we can rewrite the upper bound expression in the statement of the lemma as:

$$\sum_{h=2,..,Q}(Q-h+1)z^h(A,B) \le (Q-1-k)\sum_{h=2,..,Q} z^h(A,B) + \sum_{h=2,...k+1}(k-h+2)z^h(A,B).$$

Canceling terms and rewriting the lefthand side, we obtain

$$\sum_{h=2,...k+1}(k-h+2)z^h(A,B) + \sum_{h=k+2,..,Q}(k-h+2)z^h(A,B) \le \sum_{h=2,...k+1}(k-h+2)z^h(A,B).$$

Finally, since $\sum_{h=k+2,..,Q}(k-h+2)z^h(A,B) \le 0$ for all h and for all A, B, the result follows □

Lemma 2: For k =2,…,Q-2, the model MPQ_L implies the inequalities

$$\sum_{h=2,...k+1}(k-h+2)z^h(A,B) \le \sum_{p=0,...,k\text{-}1}\sum_{q=p+2,...,k+1}(q-p-1)x(S_p,S_q) \quad \textit{for all } \text{A,B} \subseteq V-\{1\}, A\cap B=\emptyset, k=2,...,Q\text{-}2$$
$$\textit{and all partitions } (S_0,...,S_{k+1}) \textit{ of } V \textit{ with } S_0=\{1\}, S_{k+1}=A \textit{ and } B\subseteq S_1\cup S_2.$$

$$\sum_{h=2,..,k+1}(k+h-Q)z^h(A,B) \ge \sum_{p=0,...,k\text{-}1}\sum_{q=p+2,...,k+1}(q-p-1)x(S_p,S_q) \quad \textit{for all } A,B\subseteq V-\{1\}, A\cap B=\emptyset, k=2,...,Q-2$$
$$\textit{and all partitions } (S_0,...,S_{k+1}) \textit{ of } V \textit{ with } S_0=B, S_{k+1}=\{1\} \textit{ and } B\subseteq S_{k-1}\cup S_k.$$

Proof: As before we will provide the proof for the upper bound constraints. Consider a fixed k = 2,…,Q-2 and a partition $S_0, S_1, ..,S_q,...,S_{k+1}$ of V with S_0={1}and S_{k+1}=A. The equations of the MPQ model imply the following inequalities

$$z^{h+1}(S_q,V\setminus\{1\}) = z^h(V,S_q) \qquad \text{for } q=2,...,k+1, h=1,...,Q-1. \tag{L2.1}$$

Consider P+1 mutually disjoint subsets, $Q_1,Q_2,...,Q_p,..,Q_P,T$. Consider, as well, P+2 nonnegative integers $\alpha_1,\alpha_2,...,\alpha_P,\alpha_T,\beta$ with $\beta\ge\max\{\alpha_1,\alpha_2,...,\alpha_P,\alpha_T\}\ge 0$ The previous inequalities imply that the following general inequalities are valid for all $q=2,...,k+1, h=1,..,Q-1$.

$$\alpha_1 z^{h+1}(S_q,Q_1)+\alpha_2 z^{h+1}(S_q,Q_2)+...+\alpha_p z^{h+1}(S_q,Q_P)+\alpha_T z^{h+1}(S_q,T) \le \beta z^h(V,S_q). \tag{L2.2}$$

We consider next two particular instances of this expression. In both cases, for each q=2,..,k+1, we choose h=1,..,q-1, P = k-h, Q_p = S_{h+p+1} for p =1,…,P, T = B and $\beta=q-h$. We chose the values of the multipliers as follows: For q=2,..,k , we let $\alpha_p = p$ if $p = 1,..,q-h-1$, $\alpha_p = q-h$ if p = q – h,…,P and $\alpha_T = 0$ leading to (L2.3); for q=k+1 we let $\alpha_p = p$ for $p = 1,..,P$ and $\alpha_T = q-h$ leading to (L.2.4)

$$\sum_{p=1,..,q-h-1} p\, z^{h+1}(S_q, S_{p+h+1}) + (q-h) \sum_{p=q-h,..,k-h} z^{h+1}\left(S_q, S_{l+h+1}\right) \le (q-h)\, z^h(V, S_q) \qquad \text{(L2.3)}$$
$$\text{for all } q = 2,...,k,\ h = 1,...,q-1$$

and

$$\sum_{p=1,..,k-h-1} p\, z^{h+1}(S_q, S_{p+h+1}) + (q-h) z^{h+1}\left(S_q, B\right) \le (q-h)\, z^h(V, S_q) \qquad \text{(L2.4)}$$
$$\text{for } q = k+1,\ h = 1,...,q-1.$$

We stress the relevance of the condition $B \subseteq S_1 \cup S_2$ given in the statement of the Proposition for this inequality. If this condition does not hold, the inequality (L2.4) is no longer valid, since the terms $z^{h+1}\left(S_q, B\right)$ and $z^{h+1}\left(S_q, S_{p+h+1}\right)$ for some p = 1,2, ..., k-h-1 will contain the same term $z^{h+1}\left(S_q, S_r\right)$ for some r and the multiplier of this term will equal p+(q-h) which is larger than the specified q-h . When, $B \subseteq S_1 \cup S_2$, this will not happen since the first summation starts with $z^{h+1}\left(S_q, S_3\right)$.

The two classes of inequalities, (L2.3) and (L2.4) are implied by the equalities of the MPQ_L model since they are particular cases of the more general inequality (L2.2).

Adding (L2.3) for q = 2,…,k and h=1,..q-1, adding the (L2.4) for h=1,…,k, adding the two resulting inequalities, canceling equal terms and re-indexing the summation indices, we obtain:

$$\sum_{h=2,...k+1} (k-h+2) z^h\left(A, B\right) \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \left[\sum_{p=0,..,h-1} (q-h) z^h\left(S_p, S_q\right) + \sum_{p=h,..,q-2} (q-p-1) z^h\left(S_p, S_q\right) \right].$$

The nonnegativity constraints of the variables z, imply that

$$0 \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \left[\sum_{p=0,..,h-1} (h-p-1) z^h\left(S_p, S_q\right) \right].$$

Adding the two previous inequalities gives

$$\sum_{h=2,...k+1} (k-h+2) z^h\left(A, B\right) \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \sum_{p=0,..,q-2} (q-p-1) z^h\left(S_p, S_q\right)$$

Finally, by adding the following inequality

$$0 \leq \sum_{q=2,..,k+1} \sum_{h=q,..,Q} \sum_{p=0,..,h-1} (q-p-1) z^h \left(S_p, S_q\right)$$

to the previous inequality, we obtain the desired inequality. □

Proposition 5.2. When with $S \subseteq S_1 \cup S_2$ the model MPQ_L implies the upper bound inequalities (5.6a) and when $S \subseteq S_{k-1} \cup S_k$ the MPQ_L model implies the lower bound inequalities (5.6b).

Proof: For the case k >1, the proof follows from the two previous lemmas and transitivity. Consider, the case k = 1 (as usually, we produce the proof only for the upper bound constraint). From the inequality of Lemma 1

$$f\left(A,B\right) \leq (Q-2)\,x\left(A,B\right) + z^2\left(A,B\right) \quad \textit{for all } A,B \subseteq V - \{1\}, A \cap B = \varnothing$$

and the inequality $z^2\left(A,B\right) \leq z^1\left(1,A\right)$ (this follows from the equality constraint of the MPQ model for h = 1), we obtain the desired inequality. .□

In the next result we prove the validity for k = 2.

Proposition 5.3 – The model MPQ_L implies the inequalities (5.7a/b).

Proof: As before, we present the proof for only the upper bound constraints. We consider fixed sets S' and S. By adding the first inequality of Lemma 1 for all i ∈ S' and considering k = 1, we obtain

$$\sum_{i\in S'}\sum_{j\in S} f_{ij} \leq (Q-3)\sum_{i\in S'}\sum_{j\in S} x_{ij} + \sum_{i\in S'}\sum_{j\in S}(2z_{ij}^2 + z_{ij}^3).$$

The MPQ equations for h = 1 and the constraints linking the x_{ij} with the z_{ij}^h variables imply that $\sum_{i\in S'}\sum_{j\in S} 2z_{ij}^2 \leq 2\sum_{i\in S'} x_{1i}$. For the remaining term, consider any set $P \subseteq V \setminus \{1\}$ with $S' \subseteq V\setminus(P\cup\{1\})$ and the following relationships

$$\begin{aligned}
\sum_{i\in S'}\sum_{j\in S} z_{ij}^3 &\leq \sum_{i\in S'}\sum_{k\in V\setminus\{1\}} z_{ki}^2 = \sum_{i\in S'}\sum_{k\in P} z_{ki}^2 + \sum_{i\in S'}\sum_{k\in V\setminus(P\cup\{1\})} z_{ki}^2 \\
&\leq \sum_{k\in P} z_{1k}^1 + \sum_{i\in S'}\sum_{k\in V\setminus(P\cup\{1\})} z_{ki}^2 \\
&\leq \sum_{k\in P} z_{1k}^1 + \sum_{i\in S'}\sum_{k\in V\setminus(P\cup\{1\})} z_{ki}^2 + \sum_{i\in S'}\sum_{k\in V\setminus(P\cup\{1\})}\sum_{t\geq 3} z_{ki}^t \\
&= \sum_{k\in P} z_{1k}^1 + \sum_{i\in S'}\sum_{k\in V\setminus(P\cup\{1\})}\sum_{t\geq 2} z_{ki}^t.
\end{aligned}$$

Using the constraints of the PQ formulation linking the x_{ij} with the z_{ij}^h variables, we obtain the desired inequalities. ,

To prove Proposition 5.2, we first introduce two Lemmas,

Lemma 3 – The model MPQ_L implies the following equalities

$$\sum_{u=2}^{k+1} f(S_u,S) \le \sum_{u=2}^{k+1}(Q-u)x(S_u,S) + \sum_{u=2}^{k+1}\sum_{h=2}^{u}(u-h+1)z^h(S_u,S), \quad \text{(L2.a)}$$
$$\textit{for all partitions } (S_1,S_2,...,S_{k+1},S) \textit{ of } V-\{1\}, k=1,..,Q-2$$

$$\sum_{u=2}^{k+1} f(S_u,S) \ge \sum_{u=2}^{k+1} u\ x(S_u,S) - \sum_{u=2}^{k+1}\sum_{h=Q-k+1}^{Q}(u+h-Q-1)z^h(S_u,S)\ , \quad \text{(L2.b)}$$
$$\textit{for all partitions } (S_1,S_2,...,S_{k+1},S) \textit{ of } V-\{1\}, k=1,..,Q-2.$$

Proof: The proof is similar to the proof of Lemma 1 and is omitted. ,

Lemma 4 – For k=2,…,Q-2, the model MPQ_L implies the following equalities

$$\sum_{u=2}^{k+1}\sum_{h=2}^{u}(u-h+1)z^h(S_u,B) \le \sum_{p=0}^{k-1}\sum_{q=p+2}^{k+1}(q-p-1)x(A_p,A_q) \quad k=2,...,Q-2$$
$$\textit{and partitions } (S_1,...,S_{k+1},B) \textit{ of } V\setminus\{1\} \textit{ such that } A_0=\{1\}, A_1=S_1\cup B \quad \text{(L4.a)}$$
$$\textit{and } A_u=S_u\ (u=2,...,k+1).$$

$$\sum_{u=2}^{k+1}\sum_{h=Q-k+1}^{Q}(u+h-Q-1)z^h(S_u,B) \ge \sum_{p=0}^{k-1}\sum_{q=p+2}^{k+1}(q-p-1)x(A_p,A_q) \quad k=1,...,Q-2$$
$$\textit{and partitions } (S_1,...,S_{k+1},B) \textit{ of } V\setminus\{1\} \textit{ such that } A_0=S_{k+1}, \quad \text{(L4.b)}$$
$$A_u=S_{k-u}\ (u=2,...,k-1),\ A_k=S_1\cup \mathrm{B},\ A_{k+1}=\{1\}.$$

Proof: As before, we produce a proof only for the upper bound constraint. The proof for the lower bound constraint is similar.

Recall inequalities (L.2):

$$\alpha_1\ z^{h+1}(S_q,Q_1)+\alpha_2\ z^{h+1}(S_q,Q_2)+...+\alpha_P\ z^{h+1}(S_q,Q_P)+\alpha_T\ z^{h+1}(S_q,T) \le \beta z^h(V,S_q). \quad \text{(L2.2)}$$

In the proof of Lemma 2, we have shown that the MPQ_L model implies these inequalities for all $q=2,...,k+1, h=1,...,Q-1$, with $\beta \ge \max\{\alpha_1,\alpha_2,...,\alpha_P,\alpha_T\} \ge 0$ Now, we also consider two particular instances of this expression, and, as before, in both cases, for each q=2,..,k+1, we choose h=1,..,q-1, P = k-

h, $Q_p = S_{h+p+1}$ for p =1,…,P, T = B and $\beta = q-h$. However , we now chose the values of the multipliers in a slightly different manner: For q=2,..,k , we let $\alpha_p = p$ if $p = 1,..,q-h-1$, $\alpha_p = q-h$ if p = q – h,…,P and $\alpha_T = q-h$ leading to (L4,1); for q=k+1 we let $\alpha_p = p$ for $p = 1,..,P$ and $\alpha_T = q-h$ leading to (L.4.2)

$$\sum_{p=1,..,q-h-1} p\, z^{h+1}(S_q, S_{p+h+1}) + (q-h) \sum_{p=q-h,..,k-h} z^{h+1}\left(S_q, S_{l+h+1}\right) + (q-h)\, z^{h+1}(S_q, S) \le (q-h)\, z^h(V, S_q) \qquad \text{(L4.1)}$$
$$\textit{for all } q = 2,...,k,\ h = 1,...,q-1$$

and

$$\sum_{p=1,..,k-h-1} p\, z^{h+1}(S_q, S_{p+h+1}) + (q-h) z^{h+1}\left(S_q, B\right) \le (q-h)\, z^h(V, S_q) \qquad \text{(L4.2)}$$
$$\textit{for } q = k+1,\ h = 1,...,q-1.$$

We stress the relevance of the condition $A_1 = S_1 \cup B$ specified in the statement of the Proposition, for this inequality. If this condition does not hold, the inequality (L4.2) is no longer valid, since the terms $z^{h+1}\left(S_q, B\right)$ and $z^{h+1}\left(S_q, S_{p+h+1}\right)$ for some p = 1,2, ..., k-h-1 will contain the same term $z^{h+1}\left(S_q, S_r\right)$ for some r and the multiplier of this term will equal p+(q-h) which is larger than the specified q-h . When, $B \subseteq S_1 \cup S_2$, this will not happen since the first summation starts with $z^{h+1}\left(S_q, S_3\right)$.

The two classes of inequalities, (L4.1) and (L4.2) are implied by the equalities of the MPQ_L model since they are particular cases of the more general inequality (L2.2).

Adding (L4.1) for q = 2,…,k and h=1,..q-1, adding the (L.4.2) for h=1,…,k, adding the two resulting inequalities, canceling equal terms, and re-indexing the summation indices, we obtain:

$$\sum_{u=2,...k+1} \sum_{h=2,..,u} (u-h+1) z^h\left(S_u, B\right) \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \left[\sum_{p=0,..,h-1} (q-h) z^h\left(S_p, S_q\right) + \sum_{p=h,..,q-2} \left(q-p-1\right) z^h\left(S_p, S_q\right) \right].$$

The nonnegativity constraints of the variables z, imply that

$$0 \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \left[\sum_{p=0,..,h-1} (h-p-1) z^h\left(S_p, S_q\right) \right].$$

Adding the two previous inequalities gives

$$\sum_{u=2,...k+1} \sum_{h=2,..,u} (u-h+1) z^h\left(S_u, B\right) \le \sum_{q=2,..,k+1} \sum_{h=1,..,q-1} \sum_{p=0,..,q-2} (q-p-1) z^h\left(S_p, S_q\right)$$

Finally, by adding the following inequality

$$0 \le \sum_{q=2,..,k+1} \sum_{h=q,..,Q} \sum_{p=0,..,h-1} (q-p-1) z^h \left(S_p, S_q\right)$$

to the previous inequality, we obtain the desired inequality.

Proposition 5.2 The inequalities (5.7a/b) are implied by the linear programming relaxation of MPQ.

Proof: The proof follows from the two previous lemmas and transitivity. ,

Finally, consider the inequalities

$$f_{ij} \le (Q-3)x_{ij} + 2x_{1i} + x(1,P) + x(V \setminus (P \cup \{1,j\}), i) \quad (i,j) \in A,\ i,j \ne 1, P \subseteq V \setminus \{1,i,j\} \qquad (5.8a)$$

$$f_{ij} \ge 3x_{ij} - 2x_{j1} - x(P,1) - x(j, V \setminus (P \cup \{1,i\})) \quad (i,j) \in A,\ i,j \ne 1, P \subseteq V \setminus \{1,i,j\} \qquad (5.8b)$$

Proposition 5.3: The inequalities (5.8a/b) are implied by the linear programming relaxation of the MPQ* model.

Proof: As before, we will present the proof only for the upper bound case. The proof is similar to the proof of Proposition 5.1 for the case with k = 2. In this proof we will let S' = {i} and S = {j}, and we will use constraints (3.1) instead of the MPQ equations for h > 1.

By using the Lemma 1 for k = 2, S' = {i} and S = {j}, we obtain:

$$f_{ij} \le (Q-3)x_{ij} + 2z_{ij}^2 + z_{ij}^3.$$

Using (3.1) to bound above z_{ij}^3 gives

$$z_{ij}^3 \le \sum_{k \in V \setminus \{1,i\}} z_{ki}^2 = \sum_{k \in P} z_{ki}^2 + \sum_{k \in V \setminus (P \cup \{1,i\})} z_{ki}^2.$$

Then, by using, the MPQ equations for h = 1 we obtain

$$2z_{ij}^2 \le 2z_{1i}^1$$
$$\sum_{k \in P} z_{ki}^2 \le \sum_{k \in P} z_{1k}^1$$

leading into

$$2z_{ij}^2 + z_{ij}^3 \leq 2z_{1i}^1 + \sum_{k \in P} z_{1k}^1 + \sum_{k \in V \setminus (P \cup \{1,i\})} z_{ki}^2.$$

Now, by adding the trivial inequality

$$0 \leq \sum_{h \geq 3} \sum_{k \in V \setminus (P \cup \{1,i\})} z_{ki}^h$$

and by using the equations linking the the x_{ij} variables and the z_{ij}^h variables we obtain the result. ,