On epimorphisms of ordered algebras
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A pomonoid is a quadruple \((A, \cdot, 1_A, \leq_A)\), such that \((A, \cdot, 1_A)\) is a monoid and \((A, \leq_A)\) is a poset satisfying

\[
(a_1 \leq_A a_2 \& a'_1 \leq_A a'_2) \Rightarrow a_1 \cdot a'_1 \leq_A a_2 \cdot a'_2
\]

A pomonoid homomorphism is a monotone map

\[
f : (A, \cdot, \leq_A) \rightarrow (B, \cdot, \leq_B)
\]

that is also a homomorphism of the underlying monoids.

Let us denote the category of all pomonoids and their homomorphisms by Pom.
We call $f$ an epi if it is right cancellative (in Pom), i.e.,
for every diagram (in Pom)

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{f} & \cong & \downarrow{g'} \\
\end{array}
\]

we have

\[g \circ f = g' \circ f \implies g = g'.\]
Motivation

- **Question**: Given that $f$ is an epi in Pom, is it necessarily epi in the category Mon of all monoids?
- That is, whether we must have $(g \circ f = g' \circ f \implies g = g')$ for every diagram (in Mon):

  $$
  \begin{array}{ccc}
  A & \xrightarrow{f} & B \\
  \downarrow{g} & & \downarrow{g'} \\
  & \Rightarrow & \\
  & C & \\
  \end{array}
  $$

- The answer is
Motivation

- **In one direction**
  - An immediate question is the following.
  - Do epimorphisms in other varieties (categories) of pomonoids coincide similarly with those of the underlying categories of monoids?
  - We don’t have an answer.

- **Other direction**
  - To what extent the above result can be generalized to ordered algebras vs. the (underlying) unordered algebras?
  - It is this latter direction that is the subject of this talk.
**Definition** An ordered $\Omega$-algebra is a triple $(A, \Omega, \leq_A)$ such that

- $(A, \Omega)$ is an $\Omega$-algebra,
- $(A, \leq_A)$ is a poset,
- every $f^A \in \Omega_A$ is monotone, i.e., if $f^A$ has arity $n$, then
  
  \[ \left( x_1 \leq_A x_1', \ x_2 \leq_A x_2', \ldots, x_n \leq_A x_n' \right) \rightarrow f^A(x_1, \ldots, x_n) \leq_A f^A(x_1', \ldots, x_n'). \]
A homomorphism of ordered algebras is a monotone map, that is also homomorphism of the underlying algebras.

A homomorphism $f : (A, \Omega, \leq_A) \rightarrow (B, \Omega, \leq_B)$ is called an order-embedding if $f(x) \leq_B f(x') \Rightarrow x \leq_A x'$.

**Fact** Every order-embedding is injective.

A surjective order-embedding is called an order-isomorphism.

Epimorphisms are right cancellative homomorphisms, in the sense described earlier.

**Fact** Every surjective homomorphism is an epi, but the converse is not true.
Conjecture 1

The aim is to prove or disprove the following conjecture.

**Conjecture 1** Epimorphisms in a variety of ordered algebras coincide (in the sense described above) with those of the underlying variety of unordered algebras.

We will now find a way to replace the above conjecture by one in a different context.
Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$. Then we define (an ordered subalgebra)

$$\hat{\text{Dom}}_{\mathcal{A}\mathcal{C}} = \{ x \in \mathcal{A} : \forall f, g : \mathcal{A} \rightarrow \mathcal{B}, f |_\mathcal{C} = g |_\mathcal{C} \implies f(x) = g(x) \}$$

We call $\hat{\text{Dom}}_{\mathcal{A}\mathcal{C}}$, the (ordered) dominion of $\mathcal{C}$ in $\mathcal{A}$.

Treating $\mathcal{C}$ and $\mathcal{A}$ as unordered algebras one gets the analogous definition for $\text{Dom}_{\mathcal{A}\mathcal{C}}$, the unordered dominion of $\mathcal{C}$ in $\mathcal{A}$. 
Epimorphisms and dominions, Conjecture 2

- **Fact** \( C \subseteq \text{Dom}_A C \subseteq \widehat{\text{Dom}}_A C \subseteq A. \)
- **Fact** \( f : (A, \Omega, \leq_A) \rightarrow (B, \Omega, \leq_B) \) is an epi iff \( \widehat{\text{Dom}}_B \text{Im} f = B. \)
- **Fact** \( f : (A, \Omega) \rightarrow (B, \Omega) \) is an epi iff \( \text{Dom}_B \text{Im} f = B. \)
- So Conjecture 1 will be true if
- **Conjecture 2** \( \text{Dom}_A C = \widehat{\text{Dom}}_A C, \)
  is true.
We shall next replace Conjecture 2 by yet another one.

A special amalgam of ordered algebras is a list \((C; A_1, A_2; \phi_1, \phi_2)\),

- where \(C\), \(A_1\) and \(A_2\) are ordered algebras,
- \(\phi_i : C \to A_i\), \(i \in \{1, 2\}\), are order-embeddings, and
- \(A_1\) is order-isomorphic to \(A_2\), via say \(v : A_1 \to A_2\), with \(v \circ \phi_1 = \phi_2\).

Diagrammatically:
Every special amalgam \((C; A_1, A_2; \phi_1, \phi_2)\) is **weakly** embeddable. This means the above diagram always completes to a pushout:
We say that \((\mathcal{C}; A_1, A_2; \phi_1, \phi_2)\) is **(strongly)** embeddable if the pushout is also a pullback.

\[
\begin{array}{c}
\mathcal{C} \
\downarrow \phi_2 \\
A_2 \\
\downarrow \psi_2 \\
D \\
\end{array}
\begin{array}{c}
\phi_1 \\
\rightarrow \\
A_1 \\
\psi_1 \\
\rightarrow \\
D \\
\end{array}
\]

is also a pullback.
Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$.

Take two disjoint order-isomorphic copies $\mathcal{A}_1$ and $\mathcal{A}_2$ via, say,

$$
\nu_i : \mathcal{A} \rightarrow \mathcal{A}_i, \; i \in \{1, 2\}.
$$

This gives a special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \nu_1 |_\mathcal{C}, \nu_2 |_\mathcal{C})$.

Indeed every special amalgam can be obtained in this way.
**Fact** We have

\[
\widehat{\text{Dom}_A C} \cong \text{Dom}_{A_i \nu_i |_C (C)} = \psi_i^{-1} [\psi_1(A_1) \cap \psi_2(A_2)].
\]

**Fact** The analogue of the above holds in the unordered context.

**Observation** A special amalgam \((C; A_1, A_2)\) of ordered (resp. unordered) algebras is embeddable iff

\[
\psi_i^{-1} [\psi_1(D) \cap \psi_2(D)] = \nu_i |_C (C),
\]

where \(D\) is the respective 'pushout'.

**Hence** Conjecture 2 will be true if the following holds.
Conjecture 3 A special amalgam \((C; A_1, A_2)\) is embeddable in the ordered context iff it is such in the unordered context.

Fact The last conjecture is true for semigroups (monoids) vs. ordered semigroups (monoids).

Theorem Let \(\Omega\) be a type. Then in the category of all ordered \(\Omega\)-algebras epis are surjective. (We have a written proof of this.)

Theorem Let \(\Omega\) be a type. Then in the category of all unordered \(\Omega\)-algebras epis are surjective. (We have a written proof of this, in fact this is obtained by slightly modifying the above proof.)

Corollary Conjecture 3 is true for any class of all \(\Omega\)-algebras.
An identity is called **balanced** if in both terms, that are used to define it, the number of occurrences of every variable is the same.

**Theorem** Let $\mathcal{V}$ be a variety of $\Omega$-algebras whose defining identities are balanced. Let $\mathcal{V}'$ be the variety of ordered algebras obtained from $\mathcal{V}$. Then Conjecture 3 is true for $\mathcal{V}$ vs. $\mathcal{V}'$. (We don’t have a complete written proof but we think we can write one).

**Question** What about arbitrary $\mathcal{V}$ and $\mathcal{V}'$ (We don’t have any proof, or counter example).
This research is being conducted jointly with Professor Boza Tasic. This talk was motivated by the following articles.


THANK YOU