A Markov chain on semaphore codes and the fixed point forest

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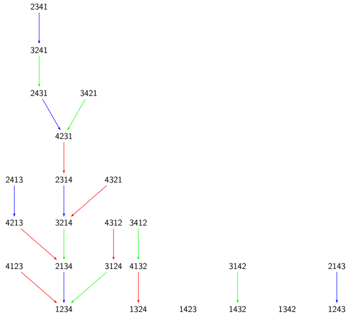
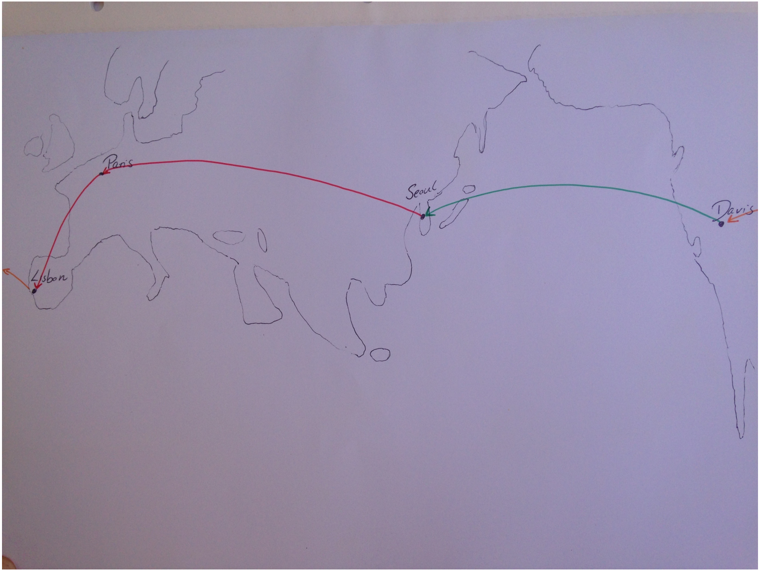
Celebrating the 60th birthday of Gracinda Gomes and Jorge Almeida
June 24, 2016



Map



Outline

1. *A Markov chain on semaphore codes*
   in joint work with John Rhodes and Pedro Silva
   arXiv:1509.03383 and arXiv:1604.00959, to appear in IJAC

2. *The fixed point forest*
   in joint work with Tobias Johnson and Erik Slivken
   arXiv:1605.09777 submitted

Appearance of *probability* and *combinatorics* in semigroup theory!

de Bruijn graph

A finite alphabet

de Bruijn graph:

vertices words in A of length k
edge $a_1 \cdots a_k \xrightarrow{a} a_2 \cdots a_k a$

random walk:

$v \xrightarrow{a} w$ with probability $\pi(a)$

transition matrix:

$\mathcal{T}_{v,w} = \pi(a)$ if $v \xrightarrow{a} w$

Stationary distribution $I\mathcal{T} = I$?
Answer: $I = (\prod_{a \in w} \pi(a))_{w \in A^k}$

Action of semigroup

Semigroup: $F(A,k) = A^1 \cup A^2 \cup \cdots \cup A^k = A^{\leqslant k}$
with product taking last k letters of concatenation

Action: $F(A,k)$ acts on A^k as

$$a_1 \ldots a_k \cdot a = a_2 \ldots a_k a \qquad \text{for } a \in A$$

Resets: elements in semigroup that act as constant maps

Here A^k

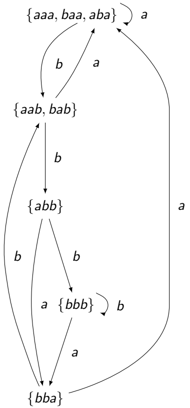
Right congruences

Motivation: Capture information that matters!

Example

$A = \{a, b\}$

$\mathrm{RC}(A^3) = \{\{aaa, baa, aba\}, \{bba\}, \{aab, bab\}, \{abb\}, \{bbb\}\}$



Right congruences

$$\mathrm{RC}(A^3) = \{\{aaa, baa, aba\}, \{bba\}, \{aab, bab\}, \{abb\}, \{bbb\}\}$$

Transition matrix:

$$\mathcal{T} = \begin{pmatrix} \pi(a) & 0 & \pi(b) & 0 & 0 \\ \pi(a) & 0 & \pi(b) & 0 & 0 \\ \pi(a) & 0 & 0 & \pi(b) & 0 \\ 0 & \pi(a) & 0 & 0 & \pi(b) \\ 0 & \pi(a) & 0 & 0 & \pi(b) \end{pmatrix}$$

Stationary distribution by lumping:

$$I = (\pi(a)^2 + \pi(a)^2\pi(b), \pi(a)\pi(b)^2, \pi(a)\pi(b), \pi(a)\pi(b)^2, \pi(b)^3)$$

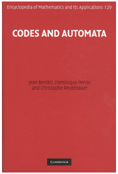
Goal: hitting time

Approach

- right congruences form a *lattice* under inclusion (meets and joins exist)
- approximation by *special congruences*
- *special congruences* $\longleftrightarrow$ semaphore codes

Suffix codes

see *Berstel, Perrin, Reutenauer* Codes and Automata



A finite alphabet
A^+ free semigroup with generators in A
A^* free monoid with generators in A

Definition

u suffix of v $\quad\Leftrightarrow\quad$ $\exists w \in A^*$ such that $wu = v$

Definition

Suffix code $\mathcal{C}$ is subset $\mathcal{C} \subseteq A^+$ such that elements in $\mathcal{C}$ are pairwise incomparable in suffix order (antichain)

Semaphore codes

Definition

A *semaphore code* is a suffix code $\mathcal{S}$ over A that has a right action:

$$u \in \mathcal{S}, a \in A \qquad \Rightarrow \qquad ua \quad \text{has suffix in } \mathcal{S}$$

Example

$\mathcal{S} = \{ba^j \mid j \geqslant 0\} = ba^*$

$ba^j \cdot a = ba^{j+1}$
$ba^j \cdot b = b$

Codes and ideals

Definition

$\mathcal{L} \subseteq A^+$ is a left ideal if $u\mathcal{L} \subseteq \mathcal{L} \; \forall u \in A^*$

suffix code = suffix minimal elements of left ideal

Definition

$\mathcal{I} \subseteq A^+$ is a ideal if $u\mathcal{I}v \subseteq \mathcal{I} \; \forall u, v \in A^*$

Connection to *semaphore codes*:
Take $u = a_j \cdots a_1 \in \mathcal{I}$. Find unique index $1 \leqslant i \leqslant j$ such that

$$a_{i-1} \cdots a_1 \notin \mathcal{I} \quad \text{but} \quad a_i \cdots a_1 \in \mathcal{I}$$

Then $a_i \cdots a_1$ is a code word.

$\mathrm{Sem}(A^k)$ set of semaphore codes with ideal in $A^{\leqslant k}$

Semaphore codes and right congruences

$u, v \in A^k$: $u \sim_{\mathcal{S}} v$ if u and v have a common suffix in $\mathcal{S}$

$\sim_{\mathcal{S}}$ defines a right congruence on A^k

Example

$A = \{a, b\}$

$$\mathcal{S} = \{aa, ab, aba, bba, abb, bbb\} \qquad \textit{semaphore code}$$

$\mathcal{S}$ yields right congruence in $\mathrm{RC}(A^3)$:

$$\{aaa, baa\}, \{aab, bab\}, \{aba\}, \{bba\}, \{abb\}, \{bbb\}$$

All congruences resulting from semaphore codes are called special right congruences $\mathrm{SRC}(A^k)$.

Approximation

$\mathrm{RC}(A^k)$ set of right congruences
$\mathrm{SRC}(A^k)$ set of special right congruences

$\mathrm{SRC}(A^k)$ full sublattice (top and bottom agree) of $\mathrm{RC}(A^k)$

Each $\rho \in \mathrm{RC}(A^k)$ has a unique largest lower (finer) approximation $\underline{\rho} \in \mathrm{SRC}(A^k)$

$$\underline{\rho} = \bigvee_{\substack{\tau \in \mathrm{SRC}(A^k) \\ \tau \subseteq \rho}} \tau \qquad \text{(join)}$$

Example

$\rho = \{\{aaa, baa, aba\}, \{bba\}, \{aab, bab\}, \{abb\}, \{bbb\}\} \in \mathrm{RC}(A^3)$
Approximation:
$\underline{\rho} = \{\{aaa, baa\}, \{aba\}, \{bba\}, \{aab, bab\}, \{abb\}, \{bbb\}\}$

Random walk on semaphore codes

probability distribution: $\pi \colon A \to [0,1]$

transition matrix: $\mathcal{T} = \sum_{a\in A} \pi(a)\mathcal{T}(a)$
with $\mathcal{T}(a)_{s,s\cdot a} = 1$ and 0 else

Theorem (RSS 2015)

Probability that word of length ℓ is reset:

$$P(\ell) = \sum_{\substack{s\in\mathcal{S}\\ \ell(s)\leqslant\ell}} \pi(s)$$

Observation

ρ and approximation $\underline{\rho}$ have same hitting time!

Random walk on semaphore codes

Example

semaphore code: $\mathcal{S} = ba^*$
resets: all words w unless $w = a^\ell$
probability that word of length 3 is reset:

$$P(3) = \pi(b) + \pi(b)\pi(a) + \pi(b)\pi(a)^2 = 1 - \pi(a)^3$$

Stationary distribution

Theorem (RSS 2015)

Stationary distribution

$$I = (\pi(s))_{s \in \mathcal{S}}$$

Transition matrix not diagonalizable

Example

$\mathcal{S} = \{a, ab, abb, bbb\}$
Jordan form

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Further work

- Semaphore codes attached to *Turing machines*
- *Profinite limits*
- Characterization of polynomial time Turing machines in this framework, including the natural semaphore codes action
- *P versus NP*??

Outline

2. *The fixed point forest*
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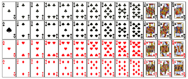
Partial Sorting Algorithm

Deck of n cards labelled $\{1, 2, \ldots, n\}$ 

Take top card and move it to slot of its value

Example

$3142 \longrightarrow 1432$

In general, view deck of cards as a permutation

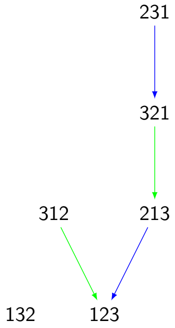
$$\pi(1)\pi(2)\ldots\pi(n) \in \mathfrak{S}_n \quad \longrightarrow \quad \pi(2)\ldots\pi(1)\ldots\pi(n)$$

Fixed Point Forest

- Each permutation eventually *sorted* to permutation with $\pi(1) = 1$.
- Opposite direction: choose *fixed point* and move it to front

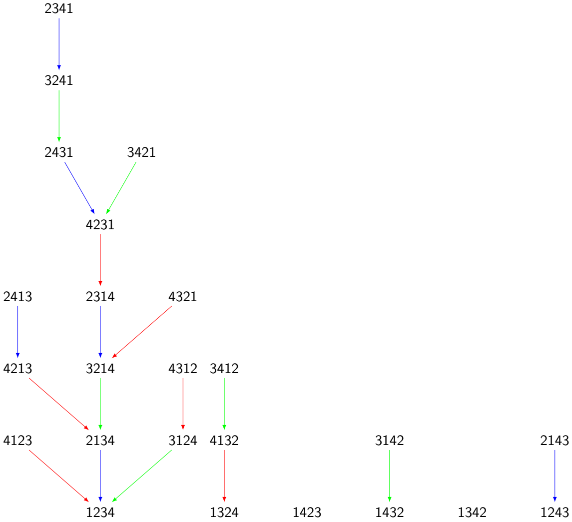
$\rightsquigarrow$ *Fixed point forest F_n* with permutation with $\pi(1) = 1$ as roots and derangements as leaves

Example (Fixed point forest F_3)



Fixed Point Forest

Example (Fixed point forest F_4)



History

Gwen McKinley UC Davis Undergraduate Thesis 2015
started as REU project at Missouri State University by Les Reid
(problem contributed by Gerhardt Hinkle)

Theorem (McKinley 2015)

- *Longest path in F_n of length $2^{n-1} - 1$ starting at $23\ldots n1$*
- *"Fractal structure"*
- *Size of tree containing $12\ldots n$ between $(n-1)!$ and $e(n-1)!$*

Open Problem

Average number of moves to root?

Goal

- Study of *local structure* of tree at random permutation π_n as $n \to \infty$
- *Stein's method*: weak convergence to tree of independent Poisson processes
- *Longest path to leaf*: geometric distribution with mean $e - 1$
- *Shortest path to leaf*: Poisson distribution with mean 1

Moving towards leaves

Recall: choose *fixed point* and move to front

Lemma

Shortest path from π_n to leaf obtained by always bumping rightmost fixed point

Example

$32415 \to 53241 \to 45321 \to 34521$
Shortest path is not unique:
$32415 \to 23415 \to 52341 \to 45231$

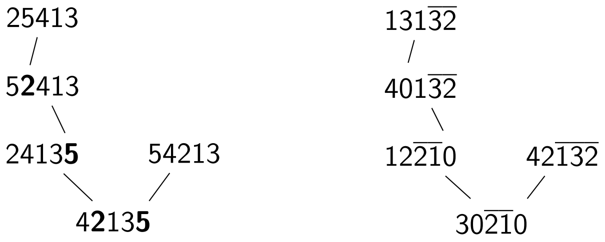
Lemma

Longest path from π_n to leaf obtained by always bumping leftmost fixed point

Remark

Longest path to leaf is unique!

Moving towards leaves in tree $T(\pi)$



Definition

$\pi \in \mathfrak{S}_n$

$\pi(i)$ is *k-separated* if $\pi(i) = i + k$

Structure of $T(\pi)$ up to level ℓ

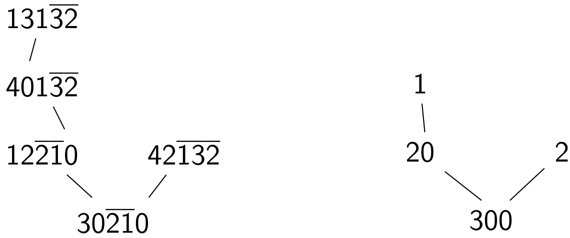
keep track of *k-separated* points for $0 \leqslant k \leqslant \ell$

or *words* in letters $0, 1, \ldots, \ell$

Limiting tree

Algorithm

- *pick* a 0 and remove
- *decrease* all letters to left of 0 by one



Remark

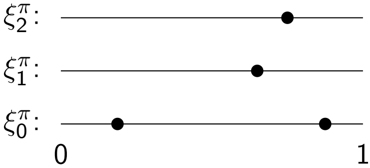
- *This forgets that* 0*-separated points in permutation at position* i *creates* $(i-1)$*-separated point.*
- *This is unlikely in limit* $n \to \infty$.

Poisson point processes

For each k, ξ_k^π represents the *k-separated points* in $[0,1]$ by rescaling by $1/n$.

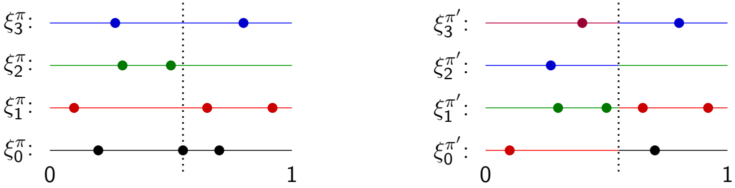
Example

$0120 \longleftrightarrow$



Bumping a fixed point

π is *abstracted permutation*
π' child given by bumping x
$\Rightarrow$ point processes $\xi_k^{\pi'}$ equals ξ_{k+1}^{π} on $[0, x)$ and ξ_k^{π} on $(x, 1]$



Results

T: tree of independent Poisson processes

Theorem (JSS 2016)

F_n weakly converges to T as $n \to \infty$

weak or *Benjamini-Schramm*: k-neighborhood of F_n converges in distribution to k-neighborhood of T

Results

L_n: length of longest path to leaf

Theorem (JSS 2016)

Distribution of L_n converges weakly to geometric distribution G with mean $e - 1$.

$$\mathbf{E}L_n^p \to \mathbf{E}G^p \qquad \forall p > 0$$

M_n: length of shortest path to leaf

Theorem (JSS 2016)

Distribution of M_n converges weakly to Poisson distribution P with mean 1.

$$\mathbf{E}M_n^p \to \mathbf{E}P^p \qquad \forall p > 0$$

Open questions

- T_n tree containing $12\ldots n$ (largest)

$$\frac{1}{n} \leqslant \mathbf{P}[\pi_n \in T_n] \leqslant \frac{e}{n},$$

Limit of $n\mathbf{P}[\pi_n \in T_n]$ as $n \to \infty$

- R_n distance from π_n to the base of its tree in the fixed point forest. *Limiting asymptotics* of $\mathbf{E}R_n$?
- *Random path* from root to leaf. Distribution of the number of steps before reaching a leaf?