Equalizers and kernels in categories of monoids

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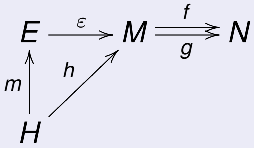
Equalizer in a full subcategory of Mon (I)

Definition

An equalizer in a full subcategory C of Mon is a morphism $\epsilon : E \to M$ satisfying $f \circ \epsilon = g \circ \epsilon$ and such that for any morphism $h : H \to M$ such that $f \circ h = g \circ h$, then there exists a unique morphism $m : H \to E$ such that the following diagram commutes:



Equalizer in a full subcategory of $\mathbf{Mon}$ (II)

- Not difficult to see that an equalizer $\epsilon : E \to M$ is a monomorphism in the category sense ($\epsilon \circ g_1 = \epsilon \circ g_2$ implies $g_1 = g_2$);
- In the categories we are considering monomorphisms are injective mappings;
- Moreover, the equalizer $\epsilon : E \to M$ of two morphisms $f, g : M \to N$ in C exists and has the form:

$$E(f, g) = \{x \in M : f(x) = g(x)\}$$

The main problem

The general problem

In a given full subcategory C of Mon, characterize $\varepsilon : E \to M$ that are equalizers.

A characterization of equalizers in Mon

We characterize the embeddings

$$\varepsilon : E \to M$$

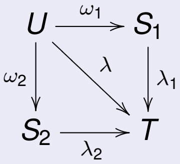
that are equalizers in Mon.

- There are three crucial notions involved in the characterization of equalizers:
  - the free product with amalgamation;
  - The submonoid $\mathrm{Dom}_M(E)$ of the elements of M dominating E;
  - The tensor product of monoids.

The free product with amalgamation (I)

Definition

A monoid amalgam is a tuple $[S_1, S_2, U; \omega_1, \omega_2]$, where $\omega_i \colon U \to S_i$ is a monomorphism for $i = 1, 2$. The amalgam is said to be *embedded* in a monoid T if there are monomorphisms $\lambda \colon U \to T$ and $\lambda_i \colon S_i \to T$ for $i = 1, 2$ such that the diagram



commutes and $\lambda_1(S_1) \cap \lambda_2(S_2) = \lambda(U)$.

The free product with amalgamation (II)

Definition

The free product with amalgamation $S_1 *_U S_2$ is the pushout of the monomorphisms $\omega_i \colon U \to S_i$, $i = 1, 2$.

$$\begin{array}{ccc} U & \xrightarrow{\omega_1} & S_1 \\ {\scriptstyle \omega_2}\downarrow & \searrow^{\lambda} & \downarrow{\scriptstyle \lambda_1} \\ S_2 & \xrightarrow[\lambda_2]{} & S_1 *_U S_2. \end{array}$$

Proposition

The amalgam is embedded in a monoid if and only if it is embedded in its free product with amalgamation.

The monoid of dominating elements

Definition (Isbell)

We say that $d \in M$ dominates E if, for all monoids N and all morphisms $f, g\colon M \to N$ in Mon, we have

$$f(u) = g(u) \text{ for every } u \in E \ \Rightarrow f(d) = g(d).$$

- Let $\mathrm{Dom}_M(E)$ be the set of all the elements $d \in M$ that dominate E.
- $\mathrm{Dom}_M(E)$ is a submonoid of M and $E \subseteq \mathrm{Dom}_M(E)$.
- If $\mathrm{Dom}_M(E) = E$, then E is said to be *closed*.

The tensor product $M \otimes_E M$

- Let X be an (M, M)-system (action of M on the left and right of X);
- Let E be a submonoid of M. $\beta\colon M \times M \to X$ is called a *bimap* if

$$\beta(mm', m'') = m\beta(m', m''),\ \beta(m, m'm'') = \beta(m, m')m''$$

$$\beta(me, m'') = \beta(m, em''),$$

for every $m, m', m'' \in M$ and $e \in E$.

The tensor product $M \otimes_E M$

Definition

A pair (P, ψ), where P is an (M, M)-system and $\psi\colon M \times M \to P$ is a bimap, is a *tensor product* of M and M over E if, for every (M, M)-system C and every bimap $\beta\colon M \times M \to C$, there is a unique (M, M)-system morphism $\beta'\colon P \to C$ such that the following diagram commutes:

$$\begin{array}{ccc} M \times M & \xrightarrow{\psi} & P \\ {\scriptstyle\beta}\downarrow & \swarrow {\scriptstyle\beta'} & \\ C & & \end{array}$$

The characterization

Theorem

The following conditions are equivalent for a submonoid E of a monoid M:

(i) *The embedding $\varepsilon\colon E \to M$ is an equalizer in the category $\mathbf{Mon}$.*

(ii) $\mathrm{Dom}_M(E) = E$.

(iii) *For any $d \in M$, $d \otimes 1 = 1 \otimes d$ in $M \otimes_E M$ if and only if $d \in E$.*

(iv) *If M' is a copy of M, then the amalgam $[M, M'; E]$ is embedded in $M *_E M'$.*

Sketch of the proof (I)

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Known results.
(i) $\Rightarrow$ (ii). Easy.
(ii) $\Rightarrow$ (i). Argument taken from Stenström. $\mathbb{Z}(M \otimes_E M)$ be the free abelian group on the tensor product $M \otimes_E M$, and let $M \times \mathbb{Z}(M \otimes_E M)$ be the monoid with operation defined by

$$(x, a)(y, b) = (xy, xb + ay)$$

and identity $(1, 0)$. Let $f \colon M \to M \times \mathbb{Z}(M \otimes_E M)$ be the monoid morphism defined by $f(x) = (x, 0)$. Let $g \colon M \to M \times \mathbb{Z}(M \otimes_E M)$ be the map defined by

$$g(x) = (x, x \otimes 1 - 1 \otimes x).$$

Sketch of the proof (II)

Then g is also a morphism, because

$$\begin{aligned} g(x)g(y) &= (x, x\otimes 1 - 1\otimes x)(y, y\otimes 1 - 1\otimes y) = \\ &= (xy, xy\otimes 1 - x\otimes y + x\otimes y - 1\otimes xy) \\ &= (xy, xy\otimes 1 - 1\otimes xy) = g(xy). \end{aligned}$$

Note that $E(f,g) = \{d \in M : f(d) = g(d)\} = \{d \in M : d\otimes 1 = 1\otimes d\}$. It is known that $d\otimes 1 = 1\otimes d$, if and only if $d \in \mathrm{Dom}_M(E)$. Hence,

$$E(f,g) = \mathrm{Dom}_M(E) = E$$

In the category of inverse monoids IMon

- From a result of Howie (67), every inverse monoid E is absolutely closed (i.e., $\mathrm{Dom}_M(E) = E$ for every monoid M containing E).
- Hence by the previous result, every inverse submonoid E of a monoid M is an equalizer in Mon.
- What about IMon? Since every amalgam of inverse monoids is embeddable in an inverse monoid, (similarly to the group case) it is possible to prove:

Proposition

In the category IMon, every monomorphism is an equalizer.

Equalizers and the coset-property (I)

- Every subgroup H of a group G is an equalizer in Grp. H satisfies the "coset"-property: $Hx \cap H \neq \emptyset$, then $x \in H$.
- What about submonoids E of a monoid M?

Definition

We say that $E \hookrightarrow M$ satisfies the right-coset condition if

$$\text{for every } m \in M,\ Em \cap E \neq \emptyset \text{ implies } m \in E$$

The left-coset condition may defined analogously.

Equalizers and the coset-property (II)

Proposition

Let E be a submonoid of a monoid M and let $\varepsilon\colon E \to M$ be the corresponding embedding. The following conditions are equivalent:

(i) *E satisfies the right coset condition.*

(ii) *$\varepsilon\colon E \to M$ is an equalizer in the category Mon and, for every $y \in E$ and $m \in M$, $y(m \otimes 1 - 1 \otimes m) = 0$ in $\mathbb{Z}(M \otimes_E M)$ implies $m \otimes 1 = 1 \otimes m$.*

Equalizers and the coset-property (III)

A simple result:

Proposition

Let $\varepsilon\colon E \to M$ be an equalizer in the category of all cancellative monoids cMon. Then E satisfies the right coset condition and the left coset condition.

Looking for the converse: By the previous result we may just deduce that if E has the right coset (or left coset) condition, then $\varepsilon\colon E \to M$ is an equalizer in Mon.

Open Problem

In the category of cancellative monoids cMon is it true that $\varepsilon\colon E \to M$ is an equalizer if and only if E satisfies the right coset and left coset condition?

Kernels in Mon

- Since we are in the category of monoids, we are in a category with zero morphisms.

Definition

The kernel of a morphism $f : M \to N$ in Mon is the equalizer of $f : M \to N$ and the zero morphism $O_{MN} : M \to N$.

- Roughly speaking the kernel of a morphism $f : M \to N$ in Mon has the form:

$$K(f) = \{x : f(x) = 1_N\}$$

Kernels in Mon

A simple fact:

Proposition

If an embedding $\varepsilon \colon E \to M$ is a kernel in Mon, then E satisfies both the right and left coset condition.

But it is not enough to characterize kernels in Mon, we need a stronger condition:

Theorem

The monomorphism $\varepsilon \colon E \to M$ is a kernel in Mon if and only if, for every $m, m' \in M$, $mEm' \cap E \neq \emptyset$ implies $mEm' \subseteq E$.

Toward a characterization of Kernels in CMon

- In Grp kernels are $\varepsilon : N \to G$ where N are normal subgroups;
- We may generalize this notion to a submonoid of a monoid M

Definition

We say that a submonoid E of M is left normal if $xE \subseteq Ex$ for all $x \in M$, and is right normal if the other inclusion $Ex \subseteq xE$ holds for all $x \in M$.

From which we may define two congruences ρ_L, ρ_R:

Proposition

The relation $y \rho_L z$ ($y \rho_R z$) if there are $u_1, u_2 \in E$ such that $u_1 y = u_2 z$ ($yu_1 = zu_2$), is a congruence.

Toward a characterization of Kernels in commutative monoids CMon

Theorem

Let E be a left normal submonoid of a monoid M. The following conditions are equivalent:

(i) $\varepsilon: E \to M$ *is an equalizer in the category* Mon *and:*

$$\forall x \in E, m \in M: \; x(m \otimes 1 - 1 \otimes m) = 0 \Rightarrow m \otimes 1 = 1 \otimes m$$

(ii) *E satisfies the right coset condition;*

(iii) $\varepsilon: E \to M$ *is a kernel in the category* Mon*;*

(iv) $E = [1]_{\rho_L} = \{ m \in M \mid \exists u \in E \text{ with } um \in E \}$*;*

Characterization of Kernels in CMon

Theorem

The following conditions are equivalent:

(i) $\varepsilon\colon E \to M$ *is a kernel in the category* CMon*;*

(ii) E *satisfies the coset condition:* $E + m \cap E \neq \emptyset$ *implies* $m \in E$*;*

(iii) $E = [1]_\rho = \{ m \in M \mid \exists u \in E \text{ with } u + m \in E \}$*;*

Open Problem

Provide a characterization of equalizers in CMon.

Divisor homomorphisms

- For commutative monoids, a divisor homomorphism is a homomorphism $f\colon M \to M'$ between two commutative monoids M, M' for which $f(x) \leq f(y)$ implies $x \leq y$ for every $x, y \in M$ ($\leq$ denotes the algebraic pre-order on M and M');
- Krull monoids are those commutative monoids M for which there exists a divisor homomorphism of M into a free commutative monoid.
- For a submonoid E of M the relation $x \leq_R y$ if $y = xu$ for some $u \in E$ is a pre-order. Dually, the relation $x \leq_L y$ if $y = ux$ for some $u \in E$ is also a pre-order.

Divisor homomorphisms (II)

Definition

Consider these preorders with $E = M$. We say that a homomorphism $f\colon M \to M'$ between two monoids M, M' is a right divisor homomorphism if $f(x) \leq_R f(y)$ implies $x \leq_R y$ for every $x, y \in M$
Similarly, for the left divisor homomorphism.

Proposition

If the monomorphism $\varepsilon\colon E \to M$ is a kernel in Mon, then ε is both a left divisor and a right divisor monomorphism.

Divisor homomorphisms (III)

Theorem

The following conditions are equivalent:

(i) $\varepsilon\colon E \to M$ *is a kernel in the category* CMon*;*

(ii) *For all* $m \in M$ $E + m \cap E \neq \emptyset$ *implies* $m \in E$*;*

(iii) $E = [1]_\rho = \{ m \in M \mid \exists u \in E \text{ with } u + m \in E \}$*;*

Furthermore, if M *is cancellative, then the previous statements are also equivalent to:*

(iv) *The monomorphism* $\varepsilon\colon E \to M$ *is a divisor monomorphism.*

The Grothendieck group

Let M be a commutative monoid. The Grothendieck group is defined as follows:

- Consider $M \times M$, and define an equivalence relation $\equiv$ on $M \times M$ setting $(x, s) \equiv (x', s')$ if $x + s' + t = x' + s + t$ for some $t \in M$;
- Let $x - s$ denote the equivalence class of (x, s) modulo the equivalence relation $\equiv$;
- Then $G(M) := M \times M/\equiv \; = \{ x - s \mid x, s \in M \}$ is the abelian group with: $(x - s) + (x' - s') = (x + x') - (s + s')$;
- There is a canonical homomorphism $f \colon M \to G(M)$, defined by $f(x) = x - 0$ for every $x \in M$, which is an embedding of monoids if and only if M is cancellative.

The category of cancellative commutative monoids $\mathbf{cCMon}$

Theorem

Let $\mathbf{cCMon}$ *be the full subcategory of* $\mathbf{CMon}$ *whose objects are all cancellative commutative monoids. Let* E *be a submonoid of a cancellative monoid* M *and* $\varepsilon \colon E \to M$ *be the embedding. Then the following conditions are equivalent:*

(i) *The monomorphism* $\varepsilon \colon E \to M$ *is a kernel in* $\mathbf{cCMon}$.

(ii) *The monomorphism* $\varepsilon \colon E \to M$ *is an equalizer in* $\mathbf{cCMon}$.

(iii) *For all* $m \in M$, $E + m \cap E \neq \emptyset$ *implies* $m \in E$.

(iv) $E = [1]_\rho = \{ m \in M \mid \exists u \in E \text{ with } u + m \in E \}$;

(v) *The monomorphism* $\varepsilon \colon E \to M$ *is a divisor monomorphism.*

(vi) *There exists a subgroup* H *of the Grothendieck group* $G(M)$ *such that* $E = M \cap H$.

The category of reduced Krull monoids $\mathbf{rKMon}$

- Krull monoids: commutative monoids for which there is a divisor homomorphism $f : M \to F$ into a free commutative monoid F;
- In the full subcategory of reduced Krull monoids: Krull monoids with trivial group of unit.

Proposition

Let $f : M \to F$ be a right (left) divisor homomorphism of a monoid M into a free monoid (free commutative monoid) F. The following conditions are equivalent:

(i) *The homomorphism f is injective.*

(ii) *The monoid M is reduced (the group of units is trivial) and cancellative.*

(iii) *The monoid M is reduced and right directly finite ($xy = y$ implies $x = 1$).*

(iv) *The monoid M is reduced.*

The category of reduced Krull monoids rKMon

Proposition

The kernel of any morphism $f\colon M \to N$ in rKMon coincides with the kernel of f in the category CMon. In particular, $E := f^{-1}(0_N)$ is a reduced Krull monoid, the embedding $\varepsilon\colon E \to M$ is a divisor homomorphism, and there exists a pure subgroup H of the free abelian group $G(M)$ such that $E = M \cap H$.

The category of free monoids FMon

Proposition

Let $\varepsilon\colon E \to M$ be an equalizer in FMon. Then E is a free submonoid of M and E is radical closed, i.e., $m \in M$ and $m^k \in E$ for some $k \geq 1$ implies $m \in E$.

Kernels are easy to characterize:

Proposition

Let M be a free monoid with free set X of generators. Then an embedding $\varepsilon\colon E \to M$ is a kernel in FMon if and only if E is a free submonoid of M generated by a subset of X.

The problem of equalizers (I)

- We say that $d \in M$ dominates E in FMon if, for all free monoids N and all morphisms $f, g\colon M \to N$, we have that $f(u) = g(u)$ for all $u \in E$ implies $f(d) = g(d)$
- Denote by $\mathrm{Dom}_{M,\mathrm{FMon}}(E)$ the set of elements that dominate U in FMon.

The problem of equalizers (II)

The only characterization we have found:

Theorem

The following conditions are equivalent:

(i) $\varepsilon\colon E \to M$ *be an equalizer in* FMon.

(ii) $\mathrm{Dom}_{M,\mathrm{FMon}}(E) = E$.

(iii) *Let* M' *be a copy of* M *and* $\varphi\colon M \to M'$ *an isomorphism. Then the amalgam* $[M, M'; E]$ *is embedded in* $M *_E M'$ *via two monomorphisms* μ, μ', *and there is a morphism* $\delta\colon M *_E M' \to N$ *into a free monoid* N *such that, for every* $x \in M$: $\delta(\mu(x)) = \delta(\mu'(\varphi(x)))$ *if and only if* $x \in E$

Equalizers in FMon

Characterize equalizer in FMon seems an hard task even for finitely generated monoids.

- Checking wether $E(f,g) \neq \{1\}$ is exactly the post correspondence problem (PCP) which is undecidable for monoids with at least 5 generators.
- It is related to the Ehrenfeucht conjecture (the Test Set Conjecture): for each language L of a finitely generated monoid M there exists a finite set $F \subseteq L$, such that for every arbitrary pair of morphisms $f, g : M \to N$ in FMon $f(x) = g(x)$ for all $x \in L$ if and only if $x \in F$ (Solved later by Guba using the Hilbert basis theorem)

Equalizers in FMon

- In the area of theoretical computer science these monoids $E(f, g)$ have been studied from a language theoretic point of view (equality languages).

Theorem (Salomaa / Culik)

Every recursively enumerable set can be expressed as an homomorphic image of the set of generators of an equalizer $E(f, g)$ in FMon*.*

Thank you!