Boolean representations of simplicial complexes: beyond matroids

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Independence

- V denotes a finite set (set of points)
- The theories of matroids and Boolean representable simplicial complexes (BRSCs) concern defining independence for a subset of V...
- ...when V is supplied with some additional structure (for example, some geometry).
- Classical example: V is a vector space over a finite field, with the usual undergraduate definition of linear independence.
- If $H \subseteq 2^V$ denotes the set of independent subsets of V, then (V, H) will consitute a (finite abstract) simplicial complex since it satisfies the axiom

(SC) $H \neq \emptyset$ and $X \subseteq Y \in H \Rightarrow X \in H$.

History

- The very developed theory of matroids was started by
  - H. Whitney, On the abstract properties of linear dependence, American Journal of Mathematics 57(3) (1935), 509–533.
- There exist many, many papers on matroids.
- The new theory of BRSCs was created in 2008 by Zur Izhakian and the author (three arXiv papers):
  - Z. Izhakian and J. Rhodes, New representations of matroids and generalizations, preprint, arXiv:1103.0503, 2011.
  - Z. Izhakian and J. Rhodes, Boolean representations of matroids and lattices, preprint, arXiv:1108.1473, 2011.
  - Z. Izhakian and J. Rhodes, C-independence and c-rank of posets and lattices, preprint, arXiv:1110.3553, 2011.

History

- The theory was developed and matured by Pedro Silva and the author in
  - J. Rhodes and P. V. Silva, Boolean Representations of Simplicial Complexes and Matroids, Springer Monographs in Mathematics, 2015.
- Further contributions have been made by Stuart Margolis, Silva and the author.

The point replacement property

- Both theories (matroids and BRSCs) satisfy the point replacement property:

(PR) For all $I, \{p\} \in H \setminus \{\emptyset\}$, there exists some $i \in I$ such that $(I \setminus \{i\}) \cup \{p\} \in H$.

- However, (PR) is too weak to get a satisfactory theory.
- (V, H) is a matroid iff it satisfies the exchange property:

(EP) For all $I, J \in H$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in H$.

- For those who know a little matroid theory: (V, H) is a matroid iff (V, H) and all its contractions satisfy (PR).

BRSCs

- We present five equivalent definitions of BRSC, five ways of defining independence.
- BRSCs satisfy axioms (SC) and (PR), and contain matroids as a particular case.

Definition 1 of BRSC

- Let $\{F_i\} \subseteq 2^V$ be nonempty.
- Let $\{G_j\}$ be the closure of $\{F_i\}$ under intersection (so each G_j is of the form $\cap_{i \in I} F_i$).
- So $\{G_j\}$ has a top element $T = V = \cap_{i \in \emptyset} F_i$ and a bottom element B (the intersection of all the F_i).

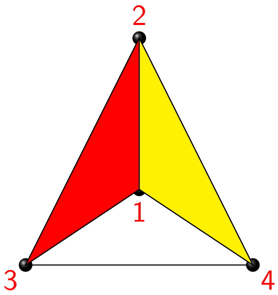
$X \subseteq V$ is independent iff there exists an enumeration $x_1, \ldots, x_n$ of the elements of X and a chain

$$G_0 \subset G_1 \subset \ldots \subset G_n$$

such that $x_j \in G_j \setminus G_{j-1}$ for $j = 1, \ldots, n$.

Example

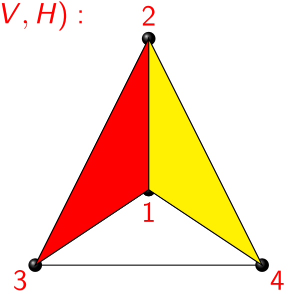
The simplicial complex (V, H) with vertex set $V = 1234$ and having $123, 124, 34$ as bases (maximal independent sets) can be depicted as



Note that (V, H) is not pure (there are bases of different size) and therefore is not a matroid.

Example

(V, H) :



Def.1: $\{F_i\} = \{1, 12, 3\}$,
$\{G_j\} = \{V, 1, 12, 3, \emptyset\}$

Definition 2 of BRSC

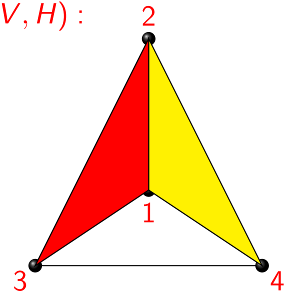
- Let $\rho : 2^V \to 2^V$ be a closure operator on the lattice $(2^V, \cup, \cap)$:
  - $X \subseteq Y \Rightarrow X\rho \subseteq Y\rho$,
  - $X \subseteq X\rho$,
  - $X\rho^2 = X\rho$.
- Write $\overline{X} = X\rho$.

$X \subseteq V$ is independent iff there exists an enumeration $x_1, \ldots, x_n$ of the elements of X such that

$$\overline{\emptyset} \subset \overline{x_1} \subset \overline{x_1 x_2} \subset \ldots \subset \overline{x_1 \ldots x_n}.$$

Example

(V, H) :



Def.1: $\{F_i\} = \{1, 12, 3\}$,
$\{G_j\} = \{V, 1, 12, 3, \emptyset\}$

Def.2: $\overline{X} = X$ if $|X| \leq 1$,
$\overline{12} = 12$,
$\overline{X} = V$ for any other $X \subseteq V$

Equivalence of 1 and 2

- Given a closure operator, the closed sets $\overline{X}$ are closed under intersection.
- Every nonempty $\{F_i\} \subseteq 2^V$ induces a closure operator on 2^V by

$$\overline{X} = \cap\{F_i \mid X \subseteq F_i\}.$$

Two remarks

- $B = \overline{\emptyset}$ consists of those points which appear in no independent set, and can therefore be omitted.
- If $p, q \in V$ are such that $\overline{p} = \overline{q}$, then pq is not independent and so we can identify p with q.

Definition 3 of BRSC

- Let (L, V) be a finite lattice sup-generated by V (i.e. each element of L is a join of elements from V).
- Canonical example: $(2^V, V)$, with union as join.

$X \subseteq V$ is independent iff there exists an enumeration $x_1, \ldots, x_n$ of the elements of X such that

$$B < x_1 < (x_1 \vee x_2) < \ldots < (x_1 \ldots x_n).$$

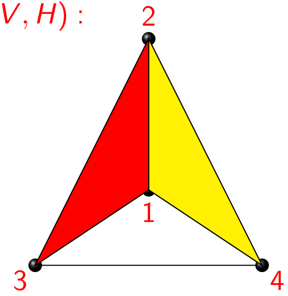
- If $\ell\!\downarrow = \{p \in V \mid p \leq \ell\}$, then this is equivalent to

$$x_i \in (x_1 \vee \ldots \vee x_i)\!\downarrow \, \setminus (x_1 \vee \ldots \vee x_{i-1})\!\downarrow$$

for $i = 1, \ldots, n$.

Example

(V, H) :



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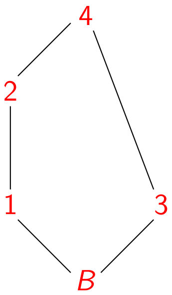
Def.3:



$4\downarrow = 1234$

$3\downarrow = 3$

$2\downarrow = 12$

$1\downarrow = 1$

$B\downarrow = \emptyset$

Equivalence of 2 and 3

- Every sup-generated lattice defines a closure operator on $(2^V, \cup, \cap)$, namely $\overline{X} = (\vee X)\downarrow$.
- If $X \mapsto \overline{X}$ is a closure operator on $(2^V, \cup, \cap)$, then its image is a lattice with join $(X \vee Y) = \overline{X \cup Y}$ and determined meet.
- E. F. Moore could have (should have) made these deductions in early 1900's.

Definition 4 of BRSC

- Let M be an $r \times |V|$ Boolean matrix (entries in $\{0, 1\}$).

$I \subseteq V = \{\text{columns of } M\}$ is independent if there exist $k = |I|$ rows $r_1, \ldots, r_k$ such that the square submatrix $N = M[r_1, \ldots, r_k; I]$ yields a lower unitriangular matrix

$$N^\pi = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ ? & 1 & 0 & \ldots & 0 \\ ? & ? & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \ldots & 1 \end{pmatrix}$$

by (independently) permuting rows and columns of N.

- If H is the set of independent subsets of V with respect to M, we say that M is a Boolean representation of (V, H).

The super Boolean semiring

- We need it to present definition 5 of a BRSC.
- A tropical algebra amusing history: what is $1 + 1$?
  - $1 + 1 = 2$ (Greek)
  - $1 + 1 = 0$ (Galois in fields of characteristic 2)
  - $1 + 1 = 1$ (Boole truth values with disjunction as sum)
  - $1 + 1 = 1^\nu = 2$ or more (super Boolean)

The super Boolean semiring

Hence the tables for the (commutative) super Boolean semiring $\mathbb{SB}$ are

$+$	0	1	1^ν
0	0	1	1^ν
1	1	1^ν	1^ν
1^ν	1^ν	1^ν	1^ν

$\cdot$	0	1	1^ν
0	0	0	0
1	0	1	1^ν
1^ν	0	1^ν	1^ν

The permanent in $\mathbb{SB}$

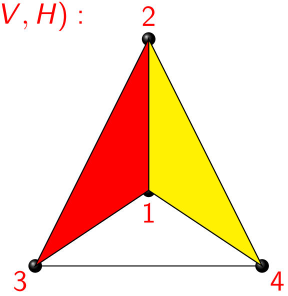
- It is a version of the determinant which omits the signs in front of each term.
- We compute the permanent $\text{per}(M)$ of a square Boolean matrix M by viewing $0, 1$ as elements of $\mathbb{SB}$.
- It is not difficult to see that $\text{per}(M) = 1$ iff we can obtain a lower unitriangular matrix by (independently) permuting rows and columns of N.
- Thus Definition 4 can be transformed to...

Definition 5 of BRSC

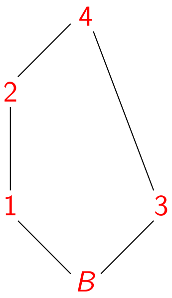
(V, H) is a BRSC iff there exists an $r \times |V|$ Boolean matrix such that H is the set of all $I \subseteq V$ such that M has a square submatrix $N = M[r_1, \ldots, r_k; I]$ with $\text{per}(N) = 1$.

Example

(V,H) :



Def.3:



$4\downarrow = 1234$

$3\downarrow = 3$

$2\downarrow = 12$

$1\downarrow = 1$

$B\downarrow = \emptyset$

Def.1: $\{F_i\} = \{1, 12, 3\}$,
$\{G_j\} = \{V, 1, 12, 3, \emptyset\}$

Def.4/5: $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

Def.2: $\overline{X} = X$ if $|X| \leq 1$,
$\overline{12} = 12$,
$\overline{X} = V$ for any other $X \subseteq V$

Remarks

- The columns $I = \{\vec{c_1}, \ldots, \vec{c_k}\} \subseteq \{0,1\}^n$ of a Boolean matrix M are independent iff

$$\lambda_1 \vec{c_1} + \ldots + \lambda_k \vec{c_k} \in \{0, 1^\nu\}^n \quad \Rightarrow \quad \lambda_1 = \ldots = \lambda_k = 0$$

  for all $\lambda_1, \ldots, \lambda_k \in \{0,1\}$.
- Standard examples of matroids are obtained by replacing the Boolean matrix M by a matrix N with coefficients over a field (finite or infinite), and then saying that I of the columns are independent iff they are independent in the usual vector space sense.
- This corresponds to Definition 5 with $\mathrm{per}(M) = 1$ replaced by $\det(N) \neq 0$.

Remarks

- A defect of matroid theory is that not all matroids are field representable (over any field).
- BRSCs remedy this: all matroids will have Boolean representations (proof: use Definition 3 with (L, V) being the geometric lattice of the matroid).
- Slightly roughly speaking, BRSCs are matroids iff *all* orderings of $I \subseteq V$ satisfy the conditions of Definitions 1–4.

Important remark

- Why are Definitions 1 and 4 equivalent?
- Roughly, given an $m \times |V|$ Boolean matrix M, consider each row r of M and let F_r be the set of columns where r is 0.
- Then $M \leftrightarrow \{F_r \mid r \text{ is a row of } M\}$ relates Definitions 4 and 1.

Examples: posets

- Let $(P, \leq)$ be a finite poset.
- For every $p \in P$, let $p\!\downarrow = \{q \in P \mid q \leq p\}$.
- Taking $\{F_i\} = \{p\!\downarrow \mid p \in P\}$ in Definition 1 of BRSC, we define independent sets of points for arbitrary posets.

Examples: algebras

- Let A be an algebraic structure.
- Let the G_j in Definition 1 be the subalgebras of A.
- Equivalently, using Definition 2 we define a closure operator by letting $\overline{X}$ be the subalgebra of A generated by $X \subseteq A$.
- Detailed examples in
  - P.J. Cameron, M. Gadouleau, J.D. Mitchell and Y. Peresse, Chains of subsemigroups, preprint, arXiv:1501.06394, 2015.
- Similarly: predicate logic structures and subgeometries.

Examples: basis in finite permutation groups

- Let G be a permutation group on the finite set V.
- Define a Galois connection

$$\begin{array}{ll} f:(2^V,\cup)\to(2^G,\cap) & g:(2^G,\cap)\to(2^V,\cup) \\ Z\mapsto \text{stabilizer of } Z & D\mapsto \text{fixed points of } D \end{array}$$

- Then $fg: 2^V \to 2^V$ is a closure operator.
- The bases of G (in the sense of Cameron) are the independent sets of the BRSC defined by fg.

Flats

- Let (V, H) be a simplicial complex.
- Then $F \subseteq V$ is a flat if

$$\text{for all } I \in H,\ I \subseteq F,\ p \in V \setminus F, \text{ we have } I \cup \{p\} \in H.$$

- We denote by $\mathrm{Fl}(V, H)$ the set of flats of (V, H).

Flats and BRSCs

$\mathrm{Fl}(V, H)$ is closed under intersection, so using Definition 1 we have

Proposition

Let (V, H) be a simplicial complex. The independent sets with respect to $\mathrm{Fl}(V, H)$ are contained in H, and the converse holds iff (V, H) is a BRSC.

Comparing representations (new idea for matroids)

- Let (V, H) be a BRSC (for instance, a matroid).
- Let $M(\mathrm{Fl}(V, H))$ be the $|\mathrm{Fl}(V, H)| \times |V|$ Boolean matrix where the 0's in each row correspond to a flat.
- Then $M(\mathrm{Fl}(V, H))$ is the largest Boolean representation of (V, H) (all others have less rows).

Comparing representations

- In general, there exist many other Boolean representations.
- In fact, the set of all Boolean representations of (V, H) constitutes a lattice (with a bottom added).
- So let us find the minimal ones (atoms of the lattice) and the minimal number of rows (mindeg).

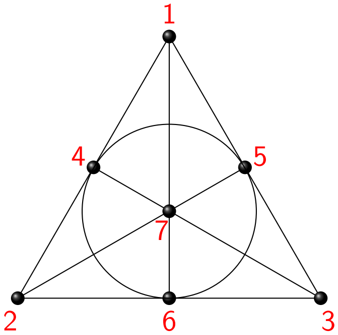
Comparing representations

- We will present a minimal representation of the Fano plane soon.
- If (V, H) is a graphic matroid, then the usual representation over $\mathbb{Z}_2$ is also a Boolean representation.

BRSCs and matroids are geometric objects

- A PEG (partial Euclidean geometry) is a finite set of points V and $\mathcal{L} \subseteq 2^V$ such that:
  - if $L \in \mathcal{L}$, then $|L| \geq 2$;
  - if $L, L' \in \mathcal{L}$ are distinct, then $|L \cap L'| \leq 1$.

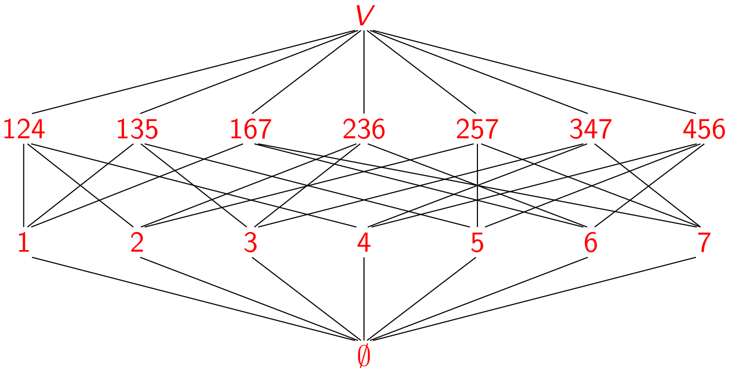
Example: the Fano plane



$$\mathcal{L} = \{124, 135, 167, 236, 257, 347, 456\}$$

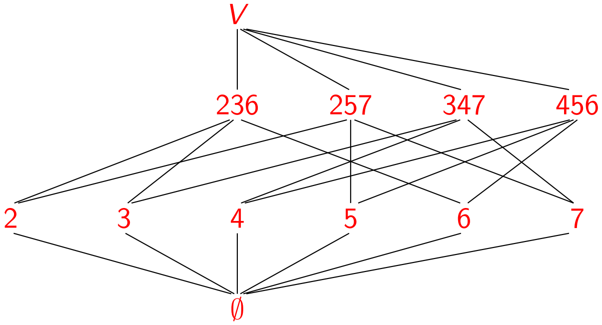
The Fano plane is the matroid defined by taking $\{F_i\} = \mathcal{L}$ in Definition 1 of BRSC.

Fano plane: the lattice of flats



This provides a Boolean representation with 7 rows corresponding to the 7 lines.

Fano plane: a minimal representation



A Boolean representation of minimum degree is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From PEGs to BRSCs

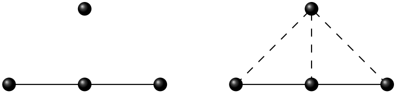
Given a PEG on V with lines $\mathcal{L}$, we say that $L \subseteq V$ is a potential line if $|L| \geq 3$ and $\mathcal{L} \cup \{L\}$ is still a PEG.
We can consider two simplicial complexes with vertex set V associated to our PEG:

(1) All subsets of V with ≤ 3 points except those 3-sets contained in some line of $\mathcal{L}$ (this is a matroid).

(2) All subsets of V with ≤ 3 points except those 3-sets contained in some line or potential line of $\mathcal{L}$ (this is a BRSC contained in the previous matroid).

FPEGs

- Now we are heading toward the great Wilson paper on combinatorics and design theory:
  - R.M. Wilson, An existence theory for pairwise balanced designs, I. Composition theorems and morphisms, J. Combinatorial Theory 13 (A) (1972), 220–245.
- We say a PEG is full (FPEG) if each pair of vertices determines a (unique) line.
- We can always embed a non full PEG into a FPEG by adding two-point lines:



PBDs

- Let $(V, \mathcal{L})$ be a FPEG and let $K = \{|L| : L \in \mathcal{L}\} \subset \{2, 3, 4, \ldots\}$.
- In design theory, this FPEG is a $PBD(|V|, K, 1)$, where
  - PBD stands for piecewise balanced design;
  - 1 means that every pair of vertices belongs to exactly 1 line, so distinct lines intersect in at most one point.
- A $PBD(v, \{k\}, 1)$ is also called a $BIBD(v, k, 1)$ (balanced incomplete block design).
- The Fano plane is a $BIBD(7, 3, 1)$.

TBRSCs

- A truncation of (V, H) is obtained by omitting all independent sets above a certain size (rank).
- Now BRSCs are not closed under truncation (in fact, every simplicial complex is the one-point contraction of some BRSC).
- But this is no problem because we can introduce the concept of TBRSCs (truncated BRSCs).
- A simplicial complex (V, H) of rank r (maximum size of an independent set) is a TBRSC if there exists an $m \times |V|$ Boolean matrix M such that the independent sets of M of rank $\leq r$ are the elements of H (but there may be independent sets of M of rank $> r$).

TBRSCs

- The theory of TBRSCs is easily developed by replacing $\mathrm{Fl}(V, H)$ by $\mathrm{TFl}(V, H)$.
- We write $F \in \mathrm{TFl}(V, H)$ if

$$\text{for all } I \in H,\ I \subseteq F,\ |I| < \mathrm{rk}(V, H),\ p \in V \setminus F, \\ \text{we have } I \cup \{p\} \in H.$$

- The theories of BRSCs and TBRSCs are similar.

The new main idea

- Consider a $PBD(v, K, 1)$ (to make it more interesting, say $2 \notin K$).
- Let (V, H) be the simple matroid (1) associated to this PBD (by omitting the 3-sets contained in some line).
- We say that $Z \subseteq V$ is a subgeometry of the matroid (V, H) if, for every pair of vertices in Z, the line determined by these vertices is also contained in Z (Wilson calls the subgeometries *closed*).

The new main idea

- But these subgeometries are precisely the elements of $\mathrm{TFl}(V, H)$.
- Thus going via Definition 1 of BRSC for the subgeometries (they form a collection of subsets closed under all intersections), they give by Definition 4 of BRSC a Boolean matrix M which yields the matroid when we truncate to rank 3.
- In general the subgeometries only define a BRSC, not a matroid.
- In this way we push the matroid into higher dimensions (the dimension being the length of the longest chain of subgeometries of the matroid).