Fiat categorification of IS_n and F_n^*

Volodymyr Mazorchuk

(Uppsala University)

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Coauthor

Joint with

Paul P. Martin (University of Leeds)

Objects of study

$n \in \{1, 2, 3, \dots\}$

$\mathbf{n} := \{1, 2, \dots, n\}$

S_n — the symmetric group on $\mathbf{n}$

IS_n — the symmetric inverse semigroup on $\mathbf{n}$ (bijections between subsets of $\mathbf{n}$)

I_n^* — the dual symmetric inverse semigroup on $\mathbf{n}$ (bijections between quotients of $\mathbf{n}$)

$F_n^* := S_n\, E(I_n^*)$ — the maximal factorizable submonoid of I_n^*

2-categories

Definition. A 2-category is a category enriched over the monoidal category **Cat** of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathscr{C}$ is given by the following data:

- objects of $\mathscr{C}$;
- small categories $\mathscr{C}(\mathtt{i},\mathtt{j})$ of morphisms;
- bifunctorial composition $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{C}(\mathtt{i},\mathtt{k})$;
- identity objects $\mathbb{1}_{\mathtt{i}} \in \mathscr{C}(\mathtt{i},\mathtt{i})$;

which are subject to the obvious set of (strict) axioms.

Terminology and the first example

Terminology:

- An object in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 1-morphism
- A morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called a 2-morphism
- Composition in $\mathscr{C}(\mathtt{i},\mathtt{j})$ is called vertical, denoted $\circ_1$
- Composition in $\mathscr{C}$ is called horizontal, denoted $\circ_0$

Principal example. The category **Cat** is a 2-category.

- Objects of **Cat** are small categories
- 1-morphisms in **Cat** are functors
- 2-morphisms in **Cat** are natural transformations
- Composition is the usual composition
- Identity 1-morphisms are identity functors

Example from semigroups

Example from ordered monoids. Let $\mathbf{S} := (S, \cdot, e, \leq)$ be an ordered monoid.

Define the 2-category $\mathscr{C}_{\mathbf{S}}$ as follows:

- $\mathscr{C}_{\mathbf{S}}$ has one (formal) object $\mathtt{i}$;
- objects in $\mathscr{C}_{\mathbf{S}}(\mathtt{i}, \mathtt{i})$ are elements in S;
- composition of 1-morphisms in $\mathscr{C}_{\mathbf{S}}$ is $\cdot$;
- the identity 1-morphism in $\mathscr{C}_{\mathbf{S}}(\mathtt{i}, \mathtt{i})$ is e;
- the set of 2-morphisms from s to t is empty if $s \not\leq t$ and consists of one element $m_{s,t}$ if $s \leq t$;
- vertical composition of 2-morphisms is the only possible map which exists due to transitivity of $\leq$;
- horizontal composition of 2-morphisms is the only possible map which exists due to admissibility of $\leq$;
- the identity 2-morphisms are $m_{s,s}$, $s \in S$.

Consequences

Consequence 1. For any monoid $(S, \cdot, e)$, the equality relation $=$ is an admissible order. This gives rise to the 2-category $\mathscr{C}_{\mathtt{S}}$, where $\mathtt{S} := (S, \cdot, e, =)$.

Consequence 2. For any inverse monoid $(S, \cdot, e, ()^{-1})$, we have the natural partial order $\prec$, which is admissible. This also gives rise to the 2-category $\mathscr{C}_{\mathbf{S}}$, where $\mathbf{S} = (S, \cdot, e, \prec)$.

Note. If S is inverse, then $\mathscr{C}_{\mathbf{S}}$ has, usually, more 2-morphisms than $\mathscr{C}_{\mathtt{S}}$.

Note. Both constructions apply to S_n, IS_n and F_n^*.

Question. What are disadvantages of these constructions.

Adjoint functors

$\mathcal{A}$, $\mathcal{C}$ — two categories

$\mathrm{F} : \mathcal{A} \to \mathcal{C}$, $\qquad \mathrm{G} : \mathcal{C} \to \mathcal{A}$ — two functors

Definition. The pair (F, G) is an adjoint pair of functors provided that

there exist $\alpha : \mathrm{Id}_{\mathcal{A}} \to \mathrm{GF}$ and $\beta : \mathrm{FG} \to \mathrm{Id}_{\mathcal{C}}$

such that

$$(\beta \circ_0 \mathrm{F}) \circ_1 (\mathrm{F} \circ_0 \alpha) = \mathrm{id}_{\mathrm{F}} \quad \text{and} \quad (\mathrm{G} \circ_0 \beta) \circ_1 (\alpha \circ_0 \mathrm{G}) = \mathrm{id}_{\mathrm{G}}.$$

Note: In **Cat** this is defined purely in terms of 2-morphisms.

2-categories with weak involution

$\mathscr{C}$ — 2-category

$\star : \mathscr{C} \to \mathscr{C}$ — weak anti-autoequivalence reversing the order of both 1-morphisms and 2-morphism

Weak: $(\mathrm{FG})^\star \cong \mathrm{G}^\star\mathrm{F}^\star$, not necessarily $(\mathrm{FG})^\star = \mathrm{G}^\star\mathrm{F}^\star$

Definition $\mathscr{C}$ is iat provides that, for each 1-morphism F, there exist 2-morphisms making $(\mathrm{F}, \mathrm{F}^\star)$ into an adjoint pair of 1-morphisms

iat: involution, adjunction, two= 2-category

Our examples: $\mathscr{S}_{S_n}$ is iat, while $\mathscr{S}_{IS_n}$ and $\mathscr{S}_{F_n^*}$ are not.

Why: Not enough 2-morphisms between 1-morphisms and the identity 1-morphism.

Main problem

Problem: Is it possible to "enlarge" $\mathscr{S}_{IS_n}$ and $\mathscr{S}_{F_n^*}$ to something iat?

Answer: YES

Rest of the talk: Construction.

Rough idea

Step 1. Start with the iat 2-category $\mathscr{S}_{S_n}$.

Step 2. Enlarge $\mathscr{S}_{S_n}$ (in different ways, depending on $\mathscr{S}_{IS_n}$ or $\mathscr{S}_{F_n^*}$) by adding new 2-morphisms.

Step 3. Linearize (e.g. over $\mathbb{Z}$ or $\mathbb{C}$).

Step 4. Split idempotents.

Enlarging $\mathscr{S}_{S_n}$ in the case of IS_n

Define a 2-category $\mathscr{A}$ as follows:

- it has one object `i`;
- its 1-morphisms are elements in S_n;
- the composition of 1-morphisms is multiplication in S_n;
- the identity 1-morphism is $\mathrm{id}_{\mathbf{n}} \in S_n$;
- for $\pi, \sigma \in S_n$, the set $\mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ is the set of all $\alpha \in \mathbf{B}_n$ (binary relations) such that $\alpha \subseteq \pi \cap \sigma$;
- for $\pi, \sigma, \tau \in S_n$, and also for $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{A}}(\sigma, \tau)$, we set $\beta \circ_1 \alpha := \beta \cap \alpha$;
- for $\pi \in S_n$, we define the identity element in $\mathrm{Hom}_{\mathscr{A}}(\pi, \pi)$ to be π;
- for $\pi, \sigma, \tau, \rho \in S_n$, and also for $\alpha \in \mathrm{Hom}_{\mathscr{A}}(\pi, \sigma)$ and $\beta \in \mathrm{Hom}_{\mathscr{A}}(\tau, \rho)$, we define $\beta \circ_0 \alpha := \beta\alpha$, the usual composition of binary relations.

What happened?

Theorem. The construct $\mathscr{A}$ is a 2-category.

Observation 1. We have $\mathrm{Hom}_{\mathscr{A}}(\pi, \sigma) = \mathrm{Hom}_{\mathscr{A}}(\sigma, \pi)$, for all π and σ, in particular, we often have many morphisms to and from identity.

Observation 2. We have $\mathrm{End}_{\mathscr{A}}(\mathrm{id}_{\mathbf{n}}) = E(IS_n)$, all binary relations which are subrelations of the identity. In particular, $\mathrm{End}_{\mathscr{A}}(\mathrm{id}_{\mathbf{n}})$ is commutative and has many idempotents.

Observation 3. $\mathscr{S}_{S_n}$ is a subcategory of $\mathscr{A}$, in particular, $\mathscr{A}$ is iat. Note that $\mathscr{S}_{S_n}$ is not 2-full in $\mathscr{A}$.

What next?

Next step 1. Linearize 2-morphisms over $\mathbb{C}$ (consider 2-morphisms as a basis for a $\mathbb{C}$-vector space and extend composition by bilinearity).

Next step 2. Take the additive closure on the level of 1-morphisms by adding formal direct sums.

Next step 3. Split idempotents on the level of 2-morphisms.

Outcome: a new 2-category, call it $\mathscr{C}$.

Properties: Finitarity: finitely many 1-morphisms up to iso and finite dimensional spaces of 2-morphisms. Inherited iat-ness.

Theorem. The 2-category $\mathscr{C}$ is fiat.

Decategorification

Grothendieck decategorification. Consider the split Grothendieck ring $[\mathscr{C}(\mathtt{i},\mathtt{i})]_\oplus$.

As abelian group: Free abelian group on $[F]$, where F is a 1-morphism, modulo $[F]=[G]+[H]$ whenever $F\cong G\oplus H$.

Ring structure: Inherited from composition of 1-morphisms.

Complexify: $\mathbb{C}\otimes_{\mathbb{Z}}[\mathscr{C}(\mathtt{i},\mathtt{i})]_\oplus$.

Main Theorem. $\mathbb{C}\otimes_{\mathbb{Z}}[\mathscr{C}(\mathtt{i},\mathtt{i})]_\oplus\cong\mathbb{C}[IS_n]$, where the basis of indecomposable 1-morphisms corresponds to the Möbius basis of $\mathbb{C}[IS_n]$ (cf. B. Steinberg. Möbius functions and semigroup representation theory. J. Combin. Theory Ser. A **113** (2006), no. 5, 866–881.)

Enlarging $\mathscr{S}_{S_n}$ in the case of F_n^*

Very similar! But:

Main difference: Definition of $\mathrm{Hom}(\pi, \sigma)$.

For this: consider S_n inside the partition monoid.

Define $\mathrm{Hom}(\pi, \sigma)$ as the set of all partitions which contain both π and σ.

Do the same as above to construct a fiat 2-category $\mathscr{D}$.

Theorem. Grothendieck decategorification of $\mathscr{D}$ is isomorphic to $\mathbb{C}[F_n^*]$.

THANK YOU!!!