Stone-type dualities for restriction semigroups

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Jorge Almeida and Gracinda Gomes

Plan of the talk

1. Overview of frames and locales and of commutative dualities.
2. Ehresmann quantal frames and quantal localic categories.
3. Restriction quantal frames, complete restriction monoids and étale localic categories.
4. Topological dualities.

GK, M. V. Lawson, A perspective on non-commutative frame theory, arXiv:1404.6516.

Frames and locales

Frames

Pointless topology studies lattices with properties similar to the properties of lattices of open sets of topological spaces.

Pointless topology studies lattices L which are

- sup-lattices: for any $x_i \in L$, $i \in I$, their join $\bigvee x_i$ exists in L.
- infinitely distributive: for any $x_i \in L$, $i \in I$, and $y \in L$

$$y \wedge (\vee_{i \in I} x_i) = \vee_{i \in I} (y \wedge x_i).$$

- Such lattices are called frames.
- A frame morphism $\varphi : F_1 \to F_2$ is required to preserve finite meets and arbitrary joins.

Locales

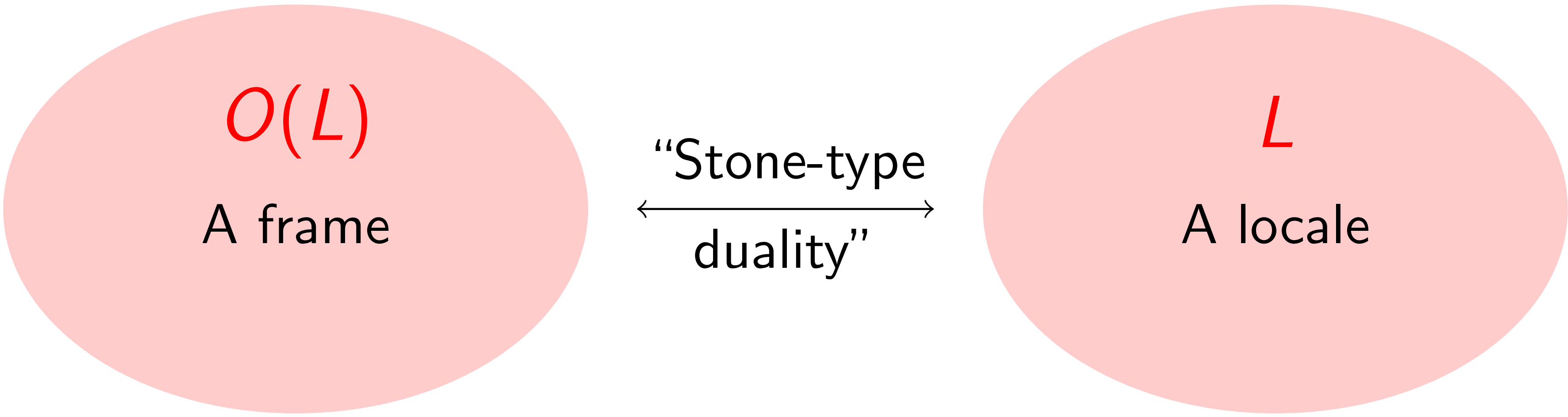
The category of locales is defined to be the opposite category to the category of frames. Locales are 'pointless topological spaces'.

Notation

If L is a locale then $O(L)$ is the frame of opens of L.

A locale morphism $\varphi : L_1 \to L_2$ is defined as the frame morphism $\varphi^* : O(L_2) \to O(L_1)$.

Frames vs locales



The adjunction

If L is a locale then points of L are defined as frame morphisms $L \to \{0, 1\}$. Topology on $\mathrm{pt}(L)$ is the subspace topology inherited from the product space $\{0, 1\}^L$.
This gives rise to the spectrum functor

$$\mathrm{pt} : \mathrm{Loc} \to \mathrm{Top}.$$

Assigning to a topological space its frame of opens leads to the functor

$$\Omega : \mathrm{Top} \to \mathrm{Loc}.$$

Theorem

The functor pt *is the right adjoint to the functor* Ω*.*

Is this adjunction an equivalence?
No!

Spatial frames and sober spaces

- A space X is sober if $\mathrm{pt}(\Omega(X)) \simeq X$.
- A locale F is spatial if $\Omega(\mathrm{pt}(F)) \simeq F$.

Theorem

The above adjunction restricts to an equivalence between the categories of spatial locales and sober spaces.

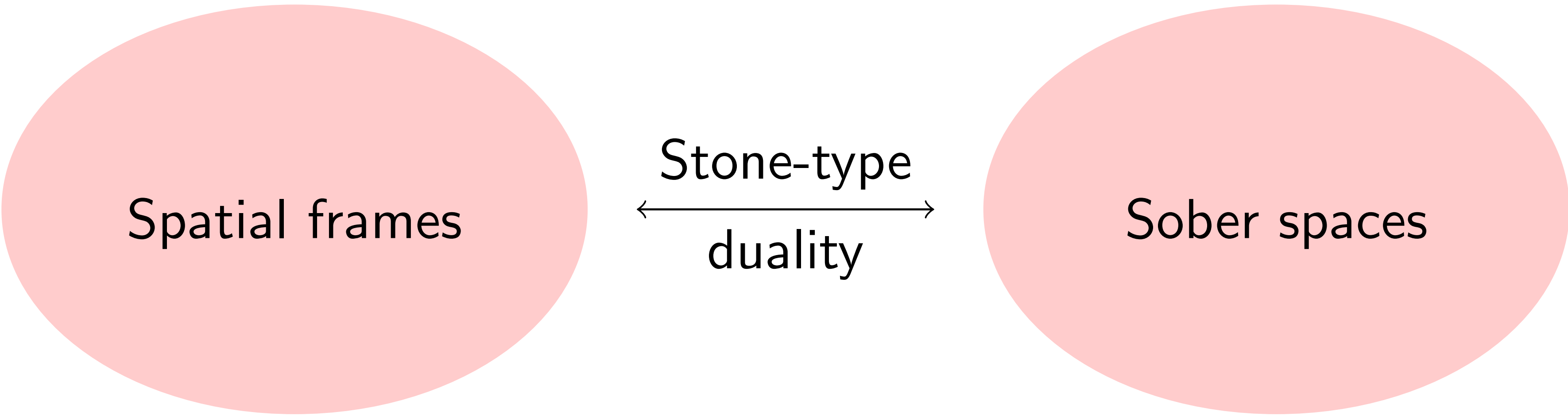
Example of a non-sober space:

$\{1, 2\}$ with indiscrete topology.

Example of non-spatial frame:

A complete non-atomic Boolean algebra, for example the Boolean algebra of Lebesgue measurable subsets of $\mathbb{R}$ modulo the ideal of sets of measure 0.

Pointset vs pointless topology



Coherent frames and distributive lattices

A space X is called spectral if it is sober and compact-open sets form a basis of the topology closed under finite intersections. A frame is called coherent if it is isomorphic to a frame of ideals of a distributive lattice.

Theorem

The following categories are pairwise equivalent:

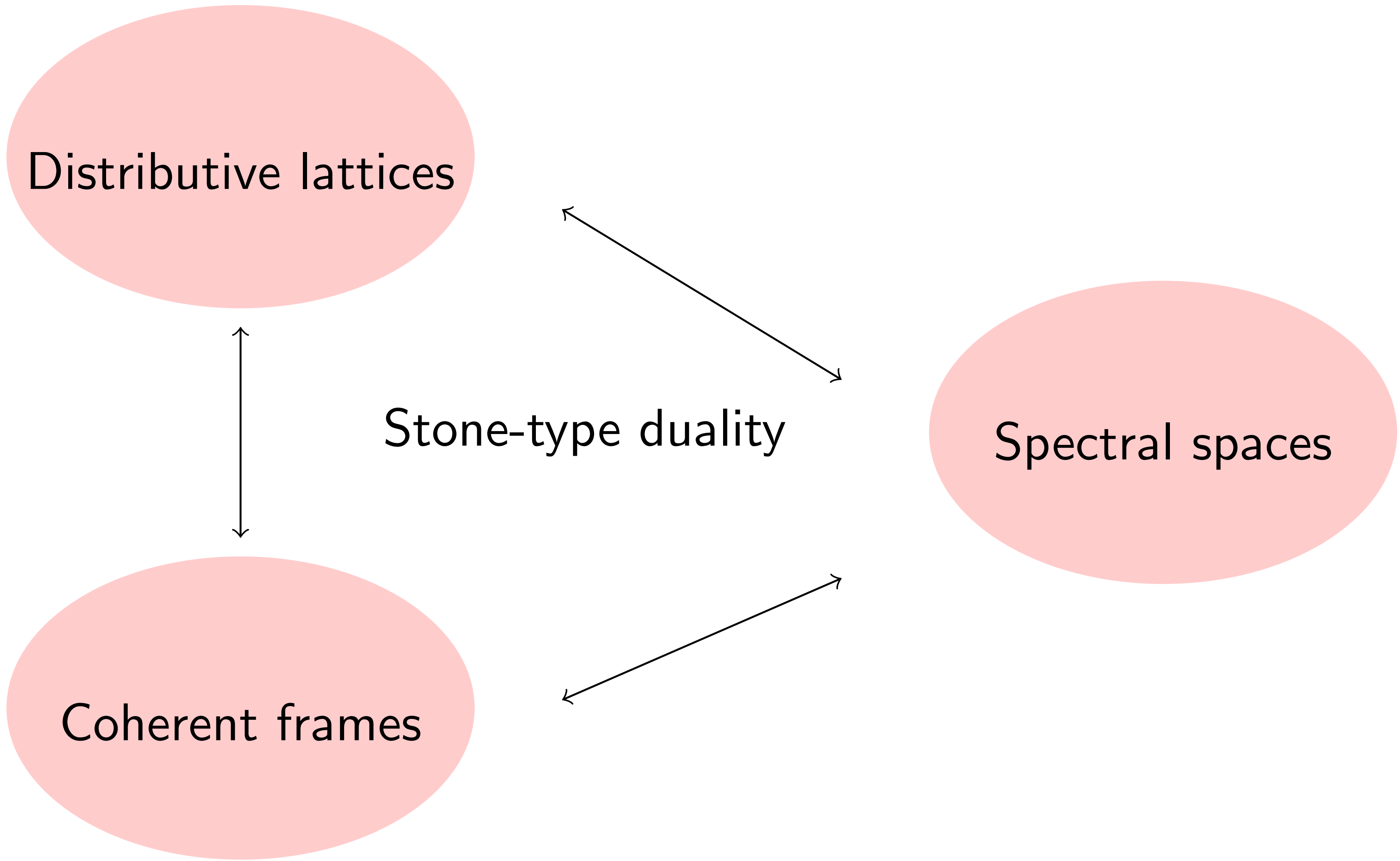
- The category of distributive lattices
- The category of coherent frames
- The opposite of the category of spectral spaces

Theorem: bounded version

The following categories are pairwise equivalent:

- The category of bounded distributive lattices
- The category of coherent frames where 1 is a finite element
- The opposite of the category of compact spectral spaces

Coherent frames and distributive lattices



Stone duality for Boolean algebras

- A locally compact Boolean space is a Hausdorff spectral space.
- A generalized Boolean algebra is a relatively complemented distributive lattice with bottom element.

Stone duality for generalized Boolean algebras

- The category of generalized Boolean algebras is dual to the category of locally compact Boolean spaces.
- The category of Boolean algebras is dual to the category of Boolean spaces.

Ehresmann quantal frames and quantal localic categories

Quantales and quantal frames

A quantale $(Q, \leq, \cdot)$ is a sup-lattice $(Q, \leq)$ equipped with a binary multiplication operation $\cdot$ such that multiplication distributes over arbitrary suprema:

$$a(\vee_{i\in I} b_i) = \vee_{i\in I}(ab_i) \text{ and } (\vee_{i\in I} b_i)a = \vee_{i\in I}(b_i a).$$

A quantale is unital if there is a multiplicative unit e and involutive, if there is an involution $*$ on Q which is a sup-lattice endomorphism.

A quantal frame is a quantale which is also a frame.

Ehresmann quantal frames

A unital quantale Q with unit e is called an Ehresmann quantale if there are two maps λ, $\rho : Q \to Q$ such that

(E1) both λ and ρ are sup-lattice endomorphisms;

(E2) if $a \leq e$ then $\lambda(a) = \rho(a) = a$;

(E3) $a = \rho(a)a$ and $a = a\lambda(a)$ for all $a \in Q$;

(E4) $\lambda(ab) = \lambda(\lambda(a)b)$, $\rho(ab) = \rho(a\rho(b))$ for all $a, b \in Q$.

Under multiplications, they are Ehresmann semigroups, introduced and first studied by Mark Lawson in 1991.

Another notation: $\lambda(a) = a^*$, $\rho(a) = a^+$.
An Ehresmann quantal frame is an Ehresmann quantale that is a also a frame.

Example

- X – non-empty set
- $A \subseteq X \times X$ – a transitive and reflexive relation
- $\mathcal{P}(A)$ – the powerset of A
- e – the identity relation

For $a \in \mathcal{P}(A)$ we define

$$a^* = \{(x,x) \in X \times X \colon \exists y \in X \text{ such that } (y,x) \in a\} \in e^{\downarrow},$$

$$a^+ = \{(y,y) \in X \times X \colon \exists x \in X \text{ such that } (y,x) \in a\} \in e^{\downarrow}.$$

$\mathcal{P}(A)$ is an Ehresmann quantal frame which generalizes the frame $e^{\downarrow} \simeq \mathcal{P}(X)$.

Localic categories

A localic category is an internal category in the category of locales.

That is, we are given the data

$$C = (C_1, C_0, u, d, r, m), \text{ or } C = (C_1, C_0), \text{ for short,}$$

where C_1 is a locale, called the locale of arrows, and C_0 is a locale, called the locale of objects, together with four locale maps

$$u\colon C_0 \to C_1, \quad d, r\colon C_1 \to C_0, \quad m\colon C_1 \times_{C_0} C_1 \to C_1,$$

called unit, domain, codomain, and multiplication, respectively. $C_1 \times_{C_0} C_1$ is the object of composable pairs defined by the pullback diagram

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle r} \\ C_1 & \xrightarrow[d]{} & C_0 \end{array}$$

Localic and topological categories

The four maps u, d, r, m are subject to conditions that express the usual axioms of a category:

1. $du = ru = id$.
2. $m(u \times id) = \pi_2$ and $m(id \times u) = \pi_1$.
3. $r\pi_1 = rm$ and $d\pi_2 = dm$.
4. $m(id \times m) = m(m \times id)$.

Topological categories are defined similarly, as internal categories in the category of topological spaces. If $C = (C_1, C_0)$ is a topological category then the space of composable pairs $C_1 \times_{C_0} C_1$ equals

$$\{(a, b) \in C_1 \times C_1 \colon d(a) = r(b)\}.$$

Ehresman quantal frames vs étale localic categories: dictionary

Commutative setting

Frame	Locale
$O(L)$	L

Non-commutative setting

Quantal frame Q	Étale localic category $C = (C_1, C_0)$
$Q = O(C_1)$	locale C_1
$e^{\downarrow} = O(C_0)$	locale C_0
quantale multiplication of Q	category multiplication of C
$*, +\colon Q \to e^{\downarrow}$	domain and range maps d and r of C
Ehresmann multiplicative: properties of $\cdot$, $*$ and $+$	quantal: properties of d and r
restriction: properties of $*$ and $+$ partial isometries generate Q	étale: properties of d and r

Adjoint pairs of maps

F_1, F_2 - frames, $f\colon F_1 \to F_2$, $g\colon F_2 \to F_1$.
f is a left adjoint of g and g a right adjoint of f if

$$f(x) \leq y \text{ iff } x \leq g(y).$$

Limit = meet, colimit = join.

RAPL = right adjoints preserve (arbitrary) limits = right adjoints preserve arbitrary joins. So if f preserves arbitrary joins, it is a right adjoint, that is, it has a left adjoint. A similar remark holds for "left adjoints preserve colimits".

Maps between locales

A locale map $f : L \to M$ is called semiopen if the defining frame map $f^* : O(M) \to O(L)$ preserves arbitrary meets. Then the left adjoint

$$f_! : O(L) \to O(M)$$

to f^* is called the direct image map of f.

f is called open if the Frobenius condition holds:

$$f_!(a \wedge f^*(b)) = f_!(a) \wedge b$$

for all $a \in O(L)$ and $b \in O(M)$.

Example: if $f : X \to Y$ is an open continuous map between topological spaces then it is open as a locale map.

The correspondence theorem

An an Ehresmann quantal frame Q is multiplicative if the right adjoint m^* of the multiplication map

$$Q \otimes_{e\downarrow} Q \to Q$$

preserves arbitrary joins and thus the multiplication map is a direct image map of a locale map.

An étale localic category $C = (C_1, C_0, u, d, r, m)$ is quantal if the maps u, d, r are open and m is semiopen (that is, $m_!$ exists and m can be 'globalized'.).

Correspondence Theorem

There is a bijective correspondence between multiplicative Ehresmann quantal frames and quantal localic categories.

Morphisms

A morphism $\varphi : Q_1 \to Q_2$ between Ehresmann quantal frames is a quantale map that is also a map of Ehresmann monoids (preserves both $*$ and $+$).
We consider the following four types of morphisms between Ehresmann quantal quantal frames:

- type 1: morphisms;
- type 2: proper morphisms (unital morphism);
- type 3: $\wedge$-morphisms (preserves non-empty finite meets);
- type 4: proper $\wedge$-morphisms (preserves all finite meets).

In the multiplicative case, morphisms between respective quantal localic categories are defined as the above morphisms but going in the opposite direction. Thus the correspondence theorem becomes a categorical duality.

Restriction quantal frames, complete restriction monoids and étale localic categories

Partial isometries

- Q – an Ehresmann quantal frame
- $a \in Q$
- a is a partial isometry if $b \leq a$ implies that $b = af = ga$ for some $f, g \leq e$
- Notation: $\mathcal{PI}(Q)$

Example

X a non-empty set, $A \subseteq X \times X$ a transitive and reflexive relation. The partial isometries of the Ehresmann quantal frame $\mathcal{P}(A)$ are precisely partial bijections.

Étale correspondence theorem

A localic category $C = (C_1, C_0)$ is étale if u, m are open and d, r are local homeomorphisms.

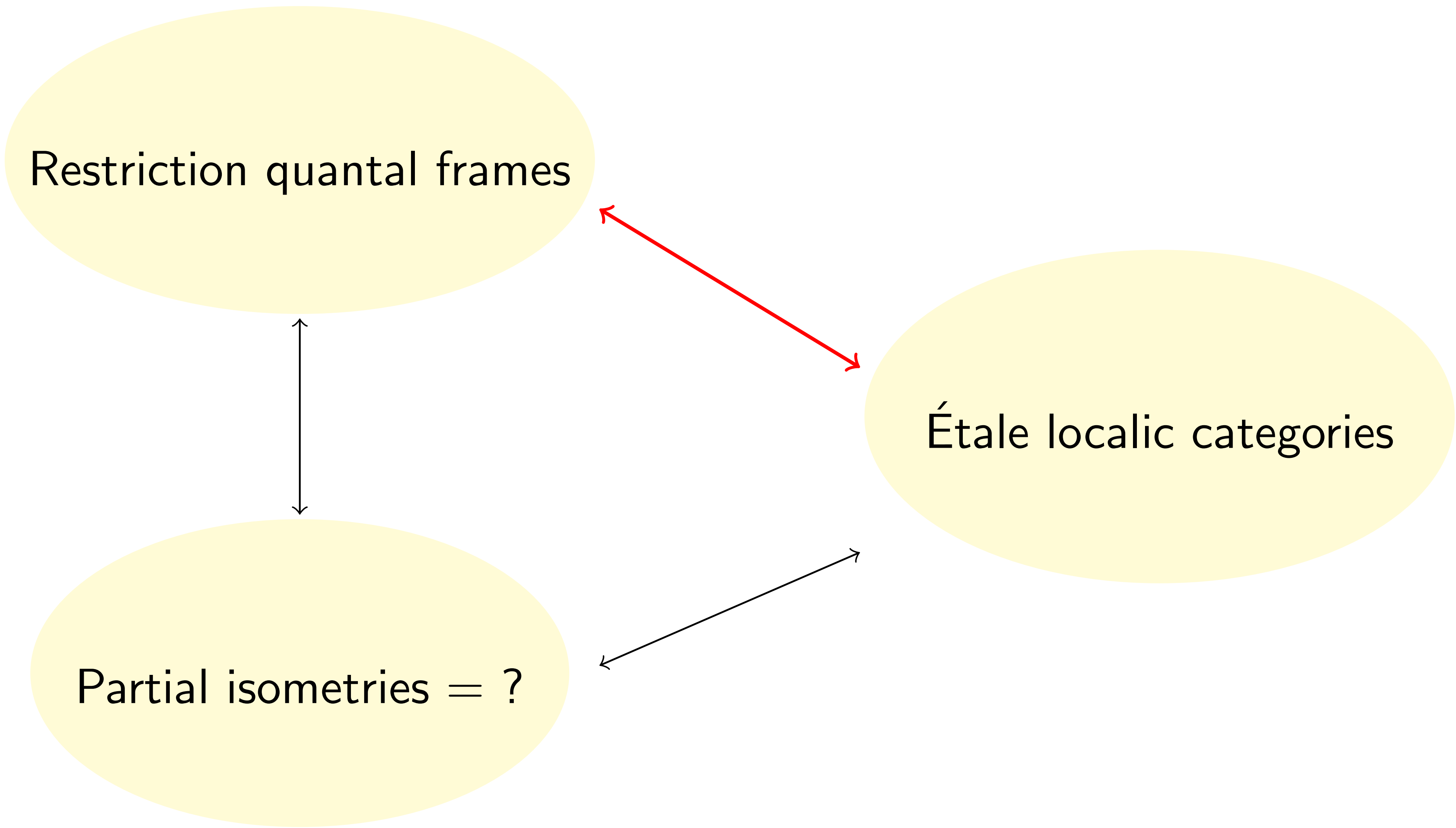
An Ehresmann quantal frame Q is a restriction quantal frame if every element is a join of partial isometries and partial isometries are closed under multiplication.

Theorem

The Correspondence Theorem restricts to the duality between restriction quantal frames and étale localic categories.

Remark: morphisms are required to preserve partial isometries! This extends and is inspired by the correspondence between inverse quantal frames and étale localic groupoids due to Pedro Resende.

Down to partial isometries



Complete restriction monoids

Restriction semigroups form a subclass of Ehresmann semigroups. They satisfy:

$$a^*b = b(ab)^*, ba^+ = (ba)^+b \text{ for all } a, b \in S.$$

Remark. Any inverse semigroup is a restriction semigroup if one defines $a^* = a^{-1}a$ and $a^+ = aa^{-1}$.

- $a, b \in S$ are compatible if $a\lambda(b) = b\lambda(a)$ and $\rho(a)b = \rho(b)a$.
- S is complete if E is a complete lattice and joins of compatible families of elements exist in S.

Equivalence with restriction quantal frames

Morphisms between complete restriction monoids

S, T – complete restriction monoids, $\varphi : S \to T$ is a morphism if

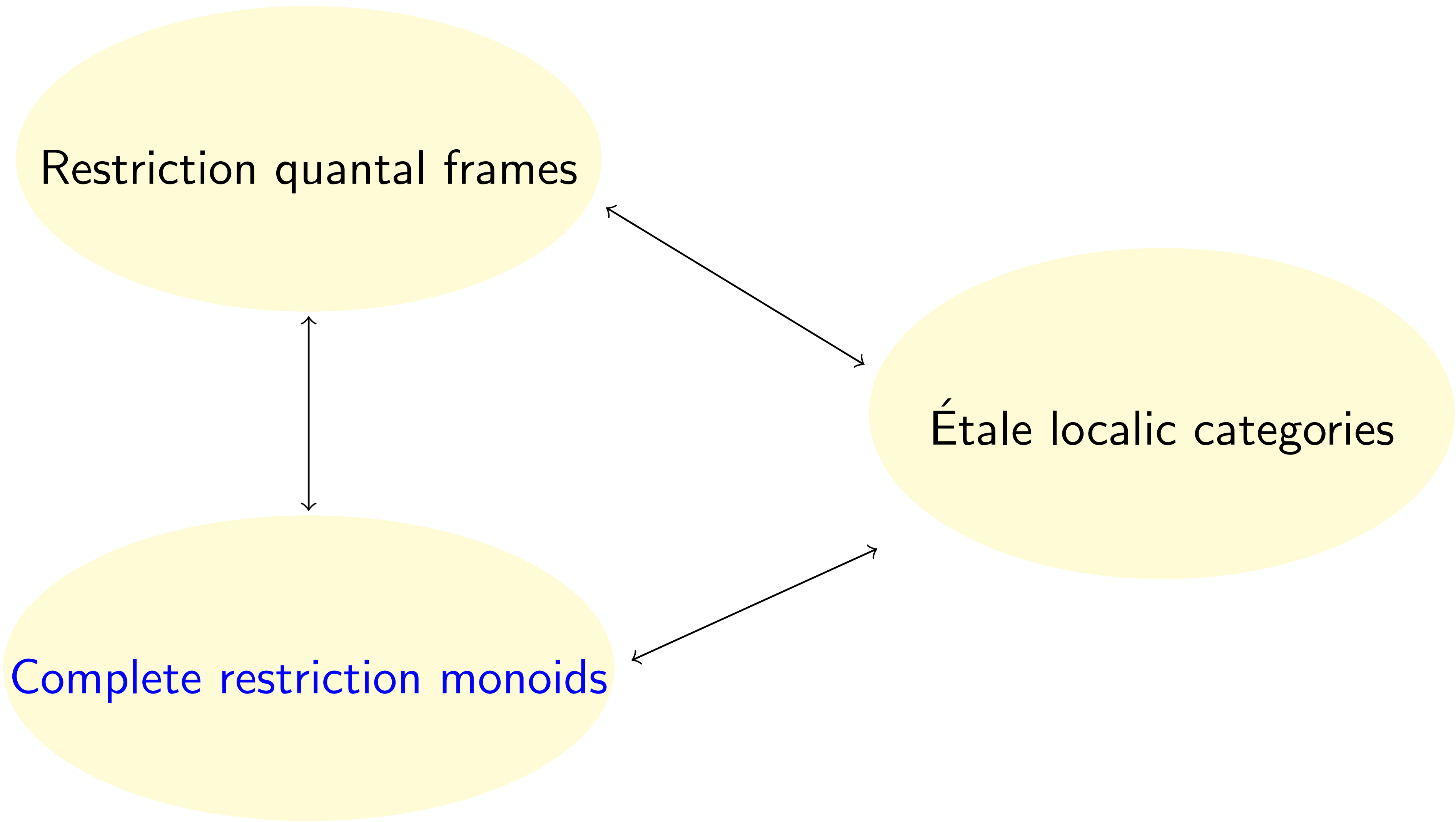
- φ is a homomorphism of restriction monoids and
- restricted to E_S, is a frame morphism from E_S to E_T.

Theorem

The category of complete restriction monoids is equivalent to the category of restriction quantal frames.

This extends an equivalence between pseudogroups and inverse quantal frames established by Pedro Resende.

The equivalences



An example

Let X be a set and

- $X \times X$ be the pair groupoid of X.
- $\mathcal{I}(X)$ be the symmetric inverse monoid on X.
- $\mathcal{P}(X \times X)$ the powerset quantale of $X \times X$.

An observation

Either of these structures allows to recover any of the other two.

Remark

This example can be generalized if instead of $X \times X$ one starts from a reflexive and transitive relation $A \subseteq X \times X$.

Topological dualities

The adjunction

Theorem

There is an adjunction between:

- the category of étale localic categories and
- the category of étale topological categories.

This adjunction is given by the spectrum and open set functors and extends the classical adjunction between locales and topological spaces.

Corollary

There is a dual adjunction between:

- the category restriction quantal frames and
- the category of étale topological categories.

This adjunction extends the classical dual adjunction between frames and topological spaces.

Sober and spatial categories

- Let $C = (C_1, C_0)$ be an étale localic category. Then the locale C_1 is spatial iff the locale C_0 is spatial. If these hold C is called spatial.
- Let $C = (C_1, C_0)$ be an étale topological category. Then the space C_1 is sober iff the space C_0 is sober. If these hold C is called sober.

Corollary

The category of spatial étale localic categories is equivalent to the category of sober étale topological categories.

Morphisms

Let $C = (C_1, C_0)$ and $D = (D_1, D_0)$ be étale topological categories. A relational covering morphism $C \to D$ is $f = (f_1, f_0)$, where $f_0 : C_0 \to D_0$ is a continuous map, $f_1 : C_1 \to \mathcal{P}(D_1)$ is a function and:

(RM1) If $b \in f_1(a)$ where $a \in C_1$ then $d(b) = f_0 d(a)$ and $r(b) = f_0 r(a)$.

(RM2) If $(a, b) \in C_1 \times_{C_0} C_1$ and $(c, d) \in D_1 \times_{D_0} D_1$ are such that $c \in f_1(a)$ and $d \in f_1(b)$ then $cd \in f_1(ab)$.

(RM3) If $d(a) = d(b)$ (or $r(a) = r(b)$) where $a, b \in C_1$ and $f_1(a) \cap f_1(b) \neq \varnothing$ then $a = b$.

(RM4) If $p = f_0(q)$ and $d(s) = p$ (resp. $r(s) = p$) where $q \in C_0$ and $s \in D_1$ then there is $t \in C_1$ such that $d(t) = q$ (resp. $r(t) = q$) and $s \in f_1(t)$.

(RM5) For any $A \in O(D_1)$:
$f_1^{-1}(A) = \{x \in C_1 \colon f_1(x) \cap A \neq \varnothing\} \in O(C_1)$.

(RM6) $uf_0(t) \in f_1 u(t)$ for any $t \in C_0$.

Types of morphisms between étale topological categories:

- Type 1: relational covering morphisms.
- Type 2: at least single-valued relational covering morphisms.
- Type 3: at most single-valued relational covering morphisms.
- Type 4: single-valued relational covering morphisms, or, equivalently, continuous covering functors.

Summary of topological dualities

RS = restriction semigroups

Algebraic object	Topological étale category $C = (C_1, C_0)$
Distributive RS	C_0 – spectral
Distributive $\wedge$ RS	C_1 (and thus also C_0) spectral
Boolean RS	C_0 – Boolean
Boolean $\wedge$ RS	C_1 (and thus also C_0) Boolean

Remark. Restriction semigroup $\to$ inverse semigroup, category $\to$ groupoid.

References

1. G. Kudryavtseva, M. V. Lawson, A perspective on non-commutative frame theory, arXiv:1404.6516.

2. M. V. Lawson, D. H. Lenz, Pseudogroups and their etale groupoids, *Adv. Math.* **244** (2013), 117–170.

3. P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.