Regular actions of groups and inverse semigroups on combinatorial structures

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(joint work with Robert Jajcay)

Group of Automorphisms of a Combinatorial Structure

Definition

- A **combinatorial structure** $(V, \mathcal{F})$ consists of a (finite) non-empty set V and a family $\mathcal{F}$ of subsets of V, $\mathcal{F} \subseteq \mathcal{P}(V)$.

  Examples include graphs, hypergraphs, geometries, designs, ...

- An **automorphism** of $(V, \mathcal{F})$ is a permutation $\varphi \in \mathbb{S}ym(V)$ satisfying the property $\varphi(B) \in \mathcal{F}$, for all $B \in \mathcal{F}$.

Classification of Automorphism Groups Problem

Given a class of combinatorial structures, classify finite groups G with the property that there exists a structure from the considered class whose full automorphism group is isomorphic to G.

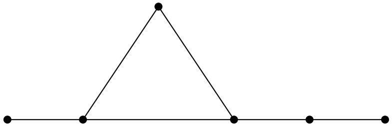
Theorem (Frucht 1939)

For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.

Proof.

- construct *any* $LC(G, X)$, $X = \{x_1, x_2, \ldots, x_k\}$
- find a family $X_1, X_2, \ldots, X_k$ of mutually non-isomorphic graphs that have no automorphisms (have a trivial automorphism group)
- replace each edge labeled x_i by the graph X_i, $1 \leq i \leq k$

□



Automorphism Groups of Graphs

Theorem (Frucht 1939)

For any finite group G there exists a graph Γ such that $Aut(\Gamma) \cong G$.

Note: We do not specify the type of action required.

Regular Group Actions

Definition
Let G be a group acting on a set V.

- The action of G on V is said to be **transitive** if for any pair of elements $u, v \in V$ there exists an element $g \in G$ such that $u^g = v$.
- The action of G on V is said to be **regular** if for any pair of elements $u, v \in V$ there exists *exactly one* element $g \in G$ such that $u^g = v$.

Regular Group Actions

Equivalently, an action of G on V is regular if

- G acts transitively on V and $Stab_G(v) = 1_G$, for all $v \in V$
- G acts transitively on V and $|G| = |V|$

Cayley Theorem

Theorem

Every group G acts regularly on itself via (left) multiplications, i.e., G is isomorphic to the group $G_L = \{\sigma_g \mid g \in G\}$ of (left) translations:

$$\sigma_g(h) = g \cdot h, \quad \textit{for all } h \in G$$

Note:

- The action of G_L on G is regular.
- Every regular action of G on a set V can be viewed as the action of G_L on G.

Regular Representations

Given a (finite) group G, find a combinatorial structure $(G, \mathcal{B})$ on G such that $Aut(G, \mathcal{B}) = G_L$.

Regular Representations

Given a (finite) group G, find a combinatorial structure $(G, \mathcal{B})$ on G such that $Aut(G, \mathcal{B}) = G_L$.

- we require an equality $Aut(G, \mathcal{B}) = G_L$

GRR

Definition

Let $\Gamma = C(G, X)$. If $Aut(\Gamma) \cong G$, then Γ is a **Graphical Regular Representation** (GRR) for G.

Cayley Graphs

Given a group G, and a generating set $X = \{x_1, x_2, \dots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain 1_G, the vertices of the **Cayley graph** $C(G, X)$ are the elements of the group G, and each vertex $g \in G$ is connected to all the vertices $gx_1, gx_2, \dots, gx_d$.

Why Cayley Graphs?

For any $g \in G$, **left-multiplication** by g is a graph automorphism of $C(G, X)$:

$$\{a, ax\} \to \{ga, gax\}$$

for all $a \in G$ and $x \in X$.

$\Longrightarrow$

$$G \leq Aut(G, X)$$

Theorem (Sabidussi)

Let Γ be a graph. Then $Aut(\Gamma)$ contains a regular group G if and only if Γ is a Cayley graph $C(G, X)$.

$\Longrightarrow$ GRR's must be Cayley graphs

Classification of Groups that Admit a GRR

Theorem (Watkins, Imrich, Godsil, ...)

Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups : $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathbb{Z}_3$, $\mathcal{Q} \times \mathbb{Z}_4$,
$\left\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \right\rangle$,
$\left\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \right\rangle$,
$\left\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \right\rangle$,
$\left\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \right\rangle$.

Proof.
About 12 papers, some of it still unpublished. □

Theorem (Babai 1980)

The finite group G admits a DRR $\overline{C}(G, X)$ if and only if G is neither the quaternion group $\mathbb{Q}_8$ nor any of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_3^2$.

Regular Representations on General Combinatorial Structures

Lemma

Let $\mathcal{I} = (V, \mathcal{B})$ *be a vertex transitive incidence structure. Then* $\mathcal{I}$ *admits a regular subgroup* G *of the full automorphism group* $Aut(\mathcal{I})$ *if and only if there exists a family of sets* $B_r \in \mathcal{P}(G)$, $1 \leq r \leq k$*, each of which contains* 1_G*, such that* $\mathcal{I}$ *is isomorphic to* $(G, \bigcup_{r=1}^{k} B_r^G)$.

Groups Admitting Regular Actions on General Combinatorial Structures

Theorem

A finite group G can be represented as a regular full automorphism group of some hypergraph if and only if G is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ *or* $\mathbb{Z}_2^2$.

The proof

- takes advantage of results concerning digraphs
- uses blocks of different sizes
- uses complements

Hypergraphs

Definition

- a pair $\mathcal{H} = (V, \mathcal{B})$, $\mathcal{B} \subseteq \mathcal{P}_k(V)$ (i.e, all the blocks are of size k), is a k-**uniform hypergraph** or simply a k-**hypergraph**
- the "usual" graph is a 2-hypergraph

Regular Representations on Hypergraphs

Theorem

A cyclic group $\mathbb{Z}_n$ can be regularly represented on a 3-hypergraph if and only if $n \neq 3, 4, 5$.

Proof.

The proof mimics the DRR:

- construct $C(\mathbb{Z}_n, \{1, -1\})$; an n-cycle
- orient all the edges the same direction (say counterclockwise)
- $Aut(\overline{C}(\mathbb{Z}_n, \{1, -1\})) = \mathbb{Z}_n$
- $\mathcal{B} = \{ \{i, i+1, i+2\} \mid 0 \leq i \leq n-1 \}$
  $\cup \ \{ \{i, i+1, i+3\} \mid 0 \leq i \leq n-1 \}$
- $Aut(\overline{C}(\mathbb{Z}_n, \{1, -1\})) = Aut(\mathbb{Z}_n, \mathcal{B})$

□

Note that cyclic groups do not admit graphical regular representation.

Open Problems

Problem

Classify finite groups G that admit a regular representation as the full automorphism group of some k-hypergraph.

Problem

For each finite group G, find all the positive integers k such that G admits a regular representation as the full automorphism group of a k-hypergraph.

Uniform Hypergraphs

Theorem

Let $n > 5$*. Then, for every* k*,* $3 \leq k \leq n-3$*, there exists a* k*-hypergraph* $\mathcal{H}_{n,k} = (\mathbb{Z}_n, \mathcal{B})$ *such that*

$$Aut(\mathcal{H}_{n,k}) = \mathbb{Z}_n$$

Proof.

$\mathcal{B} = \{ \{i, i+1, i+2, \ldots, i+k\} \mid 0 \leq i \leq n-1 \}$
$\cup \; \{ \{i, i+1, i+2, \ldots, i+(k-1), i+(k+1)\} \mid 0 \leq i \leq n-1 \}$ $\square$

Diameter

Theorem

Let $\Gamma = C(G, X)$ be a Cayley graph of G of degree $k = |X|$. If Γ admits a set $\mathcal{O}$ of $2k$ vertices non-adjacent to 1_G with the property that each vertex $g \in \mathcal{O}$ belongs to a different orbit of $Stab(1_G)$, then G admits a regular representation through a 3-hypergraph.

Corollary

Let $\Gamma = C(G, X)$ be a Cayley graph of G of degree $k = |X|$. If $diam(\Gamma) > 2k$, then G admits a regular representation through a 3-hypergraph.

Corollary

Let $r \geq 2$. All but finitely many finite groups of rank r admit regular representation through a 3-hypergraph.

Girth

Lemma

Let $\Gamma = C(G, X)$ *be a Cayley graph of valency* $|X| > k - 1$ *and girth* $g > 2k - 2$, $k \geq 2$. *Then* $Aut(C(G, X)) = Aut(G, \mathcal{B})$, *where*

$$\mathcal{B} = \{ \{g, gx, gxy\} \mid g \in G, x, y \in X \}$$

Corollary

If $\Gamma = C(G, X)$ *is a GRR for* G *of valency* $|X| > k - 1$ *and girth* $g > 2k - 2$, $k \geq 2$, *then* G *admits a regular representation through a 3-hypergraph.*

An Almost Theorem
A finite group G can be represented as a regular full automorphism group of a 3-hypergraph if and only if G is not one of the groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ or $\mathbb{Z}_2^2$.

A Conjecture
Every finite group G that has a GRR can be represented as a regular full automorphism group of some k-hypergraph for all $2 \leq k \leq |G| - 2$.

Every finite group G that can be represented as a regular full automorphism group of a 3-hypergraph can be represented as the regular full automorphism group of some k-hypergraph for all $3 \leq k \leq |G| - 3$.

Inverse semigroups of partial automorphisms

Definition

- Let $(V, \mathcal{F})$ be a combinatorial structure and U be a subset of V. The block system $\mathcal{F}'$ of the **substructure induced** by U, $(U, \mathcal{F}')$, is the system of all blocks $F \in \mathcal{F}$ that are subsets of U.
- A **partial automorphism** of a combinatorial structure $(V, \mathcal{F})$ is an isomorphism between two *induced* substructures of $(V, \mathcal{F})$, i.e., a partial bijection between two subsets $U, W \subseteq V$ that maps the induced blocks in U onto the induced blocks of W.
- The set of all partial automorphisms of $(V, \mathcal{F})$ together with the operation of partial composition forms an **inverse semigroup**; a sub-semigroup of the symmetric inverse sub-semigroup of all partial bijections from V to V.

Theorem (Wagner-Preston)

Every finite inverse semigroup is isomorphic to an inverse sub-semigroup of the symmetric inverse semigroup of all partial bijections of some finite set V.

Analogue of Cayley's theorem for groups.

Open Problems

1. Classify finite inverse semigroups that are *isomorphic* to inverse semigroups of partial automorphisms of combinatorial structures from some interesting class;
   graphs, hypergraphs, general combinatorial structures, ...

   Analogue of Frucht's theorem for groups.

2. For a specific class of representations of finite inverse semigroups classify finite inverse semigroups that admit a combinatorial structure for which the inverse semigroup of partial automorphisms is *equal to* the partial bijections from the representation.

   Analogue of GRR's for groups.

Theorem (Sieben, 2008)

The inverse semigroup of partial automorphisms of the **Cayley color graph** *of an inverse semigroup is isomorphic to the original inverse semigroup.*

Note: The inverse semigroup of partial automorphisms of a *graph* $\Gamma = (V, \mathcal{E})$ with more than one vertex is never trivial: any involution swapping two adjacent or two non-adjacent vertices is a partial automorphism of Γ.

Applications of inverse semigroups in graph theory I.

Definition
Let $\Gamma = (V, \mathcal{E})$ be a finite graph and $\mathcal{D}$ be the **deck** of Γ:
$\mathcal{D}$ is the multiset of all induced subgraphs $\Gamma - \{u\}$, $u \in V$.

Graph reconstruction conjecture (Kelly and Ulam, 1957)

Every finite graph on at least 3 vertices is **uniquely reconstructible** from its deck.

i.e., any two finite graphs that have the same decks are isomorphic.

Observation:

- For any two $u, v \in V$, the subgraphs $\Gamma - \{u\}$ and $\Gamma - \{v\}$ contain the subgraph $\Gamma - \{u, v\}$
- If the decks of $\Gamma - \{u\}$ and $\Gamma - \{v\}$ overlap in a single graph, then Γ is reconstructible
- If Γ contains a subgraph $\Gamma - \{u, v\}$ that is not isomorphic to any other subgraph $\Gamma - \{u', v'\}$, then Γ is reconstructible

  i.e., if Γ contains a subgraph $\Gamma - \{u, v\}$ for which there is no partial automorphism mapping $\Gamma - \{u, v\}$ to some $\Gamma - \{u', v'\}$, then Γ is reconstructible

Definition
Let $\Gamma = (V, \mathcal{E})$ be a finite graph. Two vertices $u, v \in V$ are **pseudo-similar** if $\Gamma - \{u\}$ and $\Gamma - \{v\}$ are isomorphic, but there exists no automorphism of Γ that would map u to v.

i.e., two vertices u and v are pseudo-similar if there exists a partial automorphism from $\Gamma - \{u\}$ and $\Gamma - \{v\}$ mapping u to v which cannot be extended into an automorphism of the whole graph.

Note: If pseudo-similar vertices did not exist, the Graph reconstruction conjecture could be easily proved.

Open problem: What is the maximal number of mutually pseudo-similar vertices in a graph of order n?

Applications of inverse semigroups in graph theory III.

Definition

A k-regular graph Γ of girth g is called a (k, g)-**cage** if Γ is of smallest possible order among all k-regular graphs of girth g.

Open problem: Does there exist a $(57, 5)$-graph of order 3250?

We do know that if the graph exists, it is not vertex-transitive, but for any two vertices u, v of such graph, there would exist a partial automorphism mapping u to v whose domain would constitute a significant part of the graph.

Most people believe the graph does not exist.

Thank you!

Všetko najlepšie, Gracinda and Jorge!