Regular actions of groups and inverse semigroups on combinatorial structures

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(joint work with Robert Jajcay)
Definition

- A **combinatorial structure** \((V, \mathcal{F})\) consists of a (finite) non-empty set \(V\) and a family \(\mathcal{F}\) of subsets of \(V\), \(\mathcal{F} \subseteq \mathcal{P}(V)\). Examples include graphs, hypergraphs, geometries, designs, ...

- An **automorphism** of \((V, \mathcal{F})\) is a permutation \(\varphi \in \text{Sym}(V)\) satisfying the property \(\varphi(B) \in \mathcal{F}\), for all \(B \in \mathcal{F}\).
Given a class of combinatorial structures, classify finite groups $G$ with the property that there exists a structure from the considered class whose full automorphism group is isomorphic to $G$. 
Automorphism Groups of Graphs

Theorem (Frucht 1939)

For any finite group $G$ there exists a graph $\Gamma$ such that $\text{Aut}(\Gamma) \cong G$.

Proof.

- construct any $LC(G, X)$, $X = \{x_1, x_2, \ldots, x_k\}$
- find a family $X_1, X_2, \ldots, X_k$ of mutually non-isomorphic graphs that have no automorphisms (have a trivial automorphism group)
- replace each edge labeled $x_i$ by the graph $X_i$, $1 \leq i \leq k$
Automorphism Groups of Graphs

Theorem (Frucht 1939)

For any finite group $G$ there exists a graph $\Gamma$ such that $\text{Aut}(\Gamma) \cong G$.

Note: We do not specify the type of action required.
Regular Group Actions

Definition
Let $G$ be a group acting on a set $V$.

- The action of $G$ on $V$ is said to be **transitive** if for any pair of elements $u, v \in V$ there exists an element $g \in G$ such that $u^g = v$.

- The action of $G$ on $V$ is said to be **regular** if for any pair of elements $u, v \in V$ there exists exactly one element $g \in G$ such that $u^g = v$. 
Equivalently, an action of $G$ on $V$ is regular if

1. $G$ acts transitively on $V$ and $\text{Stab}_G(v) = 1_G$, for all $v \in V$
2. $G$ acts transitively on $V$ and $|G| = |V|$
Theorem

Every group $G$ acts regularly on itself via (left) multiplications, i.e., $G$ is isomorphic to the group $G_L = \{\sigma_g \mid g \in G\}$ of (left) translations:

$$\sigma_g(h) = g \cdot h, \quad \text{for all } h \in G$$

Note:

- The action of $G_L$ on $G$ is regular.
- Every regular action of $G$ on a set $V$ can be viewed as the action of $G_L$ on $G$. 
Given a (finite) group $G$, find a combinatorial structure $(G, \mathcal{B})$ on $G$ such that $\text{Aut}(G, \mathcal{B}) = G_L$. 
Given a (finite) group $G$, find a combinatorial structure $(G, \mathcal{B})$ on $G$ such that $\text{Aut}(G, \mathcal{B}) = G_L$.

- we require an equality $\text{Aut}(G, \mathcal{B}) = G_L$
Definition
Let $\Gamma = C(G, X)$. If $\text{Aut}(\Gamma) \cong G$, then $\Gamma$ is a **Graphical Regular Representation** (GRR) for $G$. 
Given a group $G$, and a generating set $X = \{x_1, x_2, \ldots, x_d\}$, $\langle X \rangle = G$, that is closed under taking inverses and does not contain $1_G$, the vertices of the Cayley graph $C(G, X)$ are the elements of the group $G$, and each vertex $g \in G$ is connected to all the vertices $gx_1, gx_2, \ldots, gx_d$. 
Why Cayley Graphs?

For any $g \in G$, **left-multiplication** by $g$ is a graph automorphism of $C(G, X)$:

$$\{a, ax\} \rightarrow \{ga, gax\}$$

for all $a \in G$ and $x \in X$.

$\implies$

$$G \leq Aut(G, X)$$

Theorem (Sabidussi)

*Let $\Gamma$ be a graph. Then $Aut(\Gamma)$ contains a regular group $G$ if and only if $\Gamma$ is a Cayley graph $C(G, X)$.***

$\implies$ GRR’s must be Cayley graphs
Theorem (Watkins, Imrich, Godsil, ...) 

*Let* \( G \) *be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then* \( G \) *is an abelian group of exponent greater than 2 or* \( G \) *is a generalized dicyclic group or* \( G \) *is isomorphic to one of the 13 groups :* \( \mathbb{Z}_2^2, \mathbb{Z}_2^3, \mathbb{Z}_2^4, D_3, D_4, D_5, A_4, Q \times \mathbb{Z}_3, Q \times \mathbb{Z}_4, \)

\[
\langle a, b, c \mid a^2 = b^2 = c^2 = 1, \ abc = bca = cab \rangle,
\]

\[
\langle a, b \mid a^8 = b^2 = 1, \ b^{-1}ab = a^5 \rangle,
\]

\[
\langle a, b, c \mid a^3 = b^3 = c^2 = 1, \ ab = ba, (ac)^2 = (bc)^2 = 1 \rangle,
\]

\[
\langle a, b, c \mid a^3 = b^3 = c^3 = 1, \ ac = ca, bc = cb, b^{-1}ab = ac \rangle.
\]

**Proof.**

About 12 papers, some of it still unpublished.  

Theorem (Babai 1980)

*The finite group $G$ admits a DRR $\overline{C}(G, X)$ if and only if $G$ is neither the quaternion group $Q_8$ nor any of $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, $\mathbb{Z}_3^2$.***
Lemma

Let $\mathcal{I} = (V, \mathcal{B})$ be a vertex transitive incidence structure. Then $\mathcal{I}$ admits a regular subgroup $G$ of the full automorphism group $\text{Aut}(\mathcal{I})$ if and only if there exists a family of sets $B_r \in \mathcal{P}(G)$, $1 \leq r \leq k$, each of which contains $1_G$, such that $\mathcal{I}$ is isomorphic to $(G, \bigcup_{r=1}^k B_r^G)$. 
Theorem

A finite group $G$ can be represented as a regular full automorphism group of some hypergraph if and only if $G$ is not one of the groups $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ or $\mathbb{Z}_2^2$.

The proof

- takes advantage of results concerning digraphs
- uses blocks of different sizes
- uses complements
Definition

- a pair $\mathcal{H} = (V, \mathcal{B})$, $\mathcal{B} \subseteq \mathcal{P}_k(V)$ (i.e., all the blocks are of size $k$), is a $k$-uniform hypergraph or simply a $k$-hypergraph
- the “usual” graph is a 2-hypergraph
Theorem

A cyclic group $\mathbb{Z}_n$ can be regularly represented on a 3-hypergraph if and only if $n \neq 3, 4, 5$.

Proof.

The proof mimics the DRR:

- construct $C(\mathbb{Z}_n, \{1, -1\})$; an $n$-cycle
- orient all the edges the same direction (say counterclockwise)
- $\text{Aut}(\overline{C}(\mathbb{Z}_n, \{1, -1\})) = \mathbb{Z}_n$
- $\mathcal{B} = \{ \{i, i+1, i+2\} \mid 0 \leq i \leq n-1 \} \cup \{ \{i, i+1, i+3\} \mid 0 \leq i \leq n-1 \}$
- $\text{Aut}(\overline{C}(\mathbb{Z}_n, \{1, -1\})) = \text{Aut}(\mathbb{Z}_n, \mathcal{B})$

Note that cyclic groups do not admit graphical regular representation.
Problem
Classify finite groups $G$ that admit a regular representation as the full automorphism group of some $k$-hypergraph.

Problem
For each finite group $G$, find all the positive integers $k$ such that $G$ admits a regular representation as the full automorphism group of a $k$-hypergraph.
Theorem

Let $n > 5$. Then, for every $k$, $3 \leq k \leq n - 3$, there exists a $k$-hypergraph $\mathcal{H}_{n,k} = (\mathbb{Z}_n, \mathcal{B})$ such that

$$\text{Aut}(\mathcal{H}_{n,k}) = \mathbb{Z}_n$$

Proof.

$\mathcal{B} = \{ \{i, i + 1, i + 2, \ldots, i + k\} \mid 0 \leq i \leq n - 1 \}$

$\cup \{ \{i, i + 1, i + 2, \ldots, i + (k - 1), i + (k + 1)\} \mid 0 \leq i \leq n - 1 \}$  $\square$
Theorem
Let $\Gamma = C(G, X)$ be a Cayley graph of $G$ of degree $k = |X|$. If $\Gamma$ admits a set $\mathcal{O}$ of $2k$ vertices non-adjacent to $1_G$ with the property that each vertex $g \in \mathcal{O}$ belongs to a different orbit of $\text{Stab}(1_G)$, then $G$ admits a regular representation through a 3-hypergraph.

Corollary
Let $\Gamma = C(G, X)$ be a Cayley graph of $G$ of degree $k = |X|$. If $\text{diam}(\Gamma) > 2k$, then $G$ admits a regular representation through a 3-hypergraph.

Corollary
Let $r \geq 2$. All but finitely many finite groups of rank $r$ admit regular representation through a 3-hypergraph.
Lemma
Let $\Gamma = C(G, X)$ be a Cayley graph of valency $|X| > k - 1$ and girth $g > 2k - 2$, $k \geq 2$. Then $\text{Aut}(C(G, X)) = \text{Aut}(G, \mathcal{B})$, where

$$\mathcal{B} = \{ \{g, gx, gxy\} | g \in G, x, y \in X \}$$

Corollary
If $\Gamma = C(G, X)$ is a GRR for $G$ of valency $|X| > k - 1$ and girth $g > 2k - 2$, $k \geq 2$, then $G$ admits a regular representation through a 3-hypergraph.
An Almost Theorem
A finite group $G$ can be represented as a regular full automorphism group of a 3-hypergraph if and only if $G$ is not one of the groups $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ or $\mathbb{Z}_2^2$.

A Conjecture
Every finite group $G$ that has a GRR can be represented as a regular full automorphism group of some $k$-hypergraph for all $2 \leq k \leq |G| - 2$.

Every finite group $G$ that can be represented as a regular full automorphism group of a 3-hypergraph can be represented as the regular full automorphism group of some $k$-hypergraph for all $3 \leq k \leq |G| - 3$. 
Definition

- Let \((V, \mathcal{F})\) be a combinatorial structure and \(U\) be a subset of \(V\). The block system \(\mathcal{F}'\) of the substructure induced by \(U\), \((U, \mathcal{F}')\), is the system of all blocks \(F \in \mathcal{F}\) that are subsets of \(U\).

- A partial automorphism of a combinatorial structure \((V, \mathcal{F})\) is an isomorphism between two induced substructures of \((V, \mathcal{F})\), i.e., a partial bijection between two subsets \(U, W \subseteq V\) that maps the induced blocks in \(U\) onto the induced blocks of \(W\).

- The set of all partial automorphisms of \((V, \mathcal{F})\) together with the operation of partial composition forms an inverse semigroup; a sub-semigroup of the symmetric inverse sub-semigroup of all partial bijections from \(V\) to \(V\).
Theorem (Wagner-Preston)

Every finite inverse semigroup is isomorphic to an inverse sub-semigroup of the symmetric inverse semigroup of all partial bijections of some finite set $V$.

Analogue of Cayley’s theorem for groups.
Open Problems

1. Classify finite inverse semigroups that are *isomorphic* to inverse semigroups of partial automorphisms of combinatorial structures from some interesting class; graphs, hypergraphs, general combinatorial structures, ...

   Analogue of Frucht’s theorem for groups.

2. For a specific class of representations of finite inverse semigroups classify finite inverse semigroups that admit a combinatorial structure for which the inverse semigroup of partial automorphisms is *equal to* the partial bijections from the representation.

   Analogue of GRR’s for groups.
Theorem (Sieben, 2008)

*The inverse semigroup of partial automorphisms of the** Cayley color graph **of an inverse semigroup is isomorphic to the original inverse semigroup.*

**Note:** The inverse semigroup of partial automorphisms of a graph $\Gamma = (V, E)$ with more than one vertex is never trivial: any involution swapping two adjacent or two non-adjacent vertices is a partial automorphism of $\Gamma$.

\[ u \leftrightarrow v \quad \text{and} \quad u \leftrightarrow v \]
Definition
Let \( \Gamma = (V, E) \) be a finite graph and \( D \) be the deck of \( \Gamma \): \( D \) is the multiset of all induced subgraphs \( \Gamma - \{u\}, \ u \in V \).

Graph reconstruction conjecture (Kelly and Ulam, 1957)

Every finite graph on at least 3 vertices is uniquely reconstructible from its deck.

i.e., any two finite graphs that have the same decks are isomorphic.
Observation:

- For any two \( u, v \in V \), the subgraphs \( \Gamma - \{u\} \) and \( \Gamma - \{v\} \) contain the subgraph \( \Gamma - \{u, v\} \).

- If the decks of \( \Gamma - \{u\} \) and \( \Gamma - \{v\} \) overlap in a single graph, then \( \Gamma \) is reconstructible.

- If \( \Gamma \) contains a subgraph \( \Gamma - \{u, v\} \) that is not isomorphic to any other subgraph \( \Gamma - \{u', v'\} \), then \( \Gamma \) is reconstructible.

i.e., if \( \Gamma \) contains a subgraph \( \Gamma - \{u, v\} \) for which there is no partial automorphism mapping \( \Gamma - \{u, v\} \) to some \( \Gamma - \{u', v'\} \), then \( \Gamma \) is reconstructible.
Definition
Let $\Gamma = (V, E)$ be a finite graph. Two vertices $u, v \in V$ are **pseudo-similar** if $\Gamma - \{u\}$ and $\Gamma - \{v\}$ are isomorphic, but there exists no automorphism of $\Gamma$ that would map $u$ to $v$.
i.e., two vertices $u$ and $v$ are pseudo-similar if there exists a partial automorphism from $\Gamma - \{u\}$ and $\Gamma - \{v\}$ mapping $u$ to $v$ which cannot be extended into an automorphism of the whole graph.

**Note:** If pseudo-similar vertices did not exist, the Graph reconstruction conjecture could be easily proved.

**Open problem:** What is the maximal number of mutually pseudo-similar vertices in a graph of order $n$?
Definition
A $k$-regular graph $\Gamma$ of girth $g$ is called a $(k, g)$-cage if $\Gamma$ is of smallest possible order among all $k$-regular graphs of girth $g$.

Open problem: Does there exist a $(57, 5)$-graph of order 3250?

We do know that if the graph exists, it is not vertex-transitive, but for any two vertices $u, v$ of such graph, there would exist a partial automorphism mapping $u$ to $v$ whose domain would constitute a significant part of the graph.

Most people believe the graph does not exist.
Thank you!
Všetko najlepšie, Gracinda and Jorge!