Reducibility of pseudovarieties of the form V*D

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International Conference on Semigroups and Automata 2016
Celebrating the 60th birthday of Jorge Almeida and Gracinda Gomes
Lisbon, June 21, 2016

- Motivation and context
- Introduction
- Contributions

At the *Jam Session on Semigroups and Automata* that took place at CMUP in 2011, J. Almeida proposed the following

Question

Is the pseudovariety **LG** *tame?*

As one recalls,

- **G** is the pseudovariety of all finite *groups*.
- **LG** is the pseudovariety of all finite *local groups*, that is, finite semigroups S such that $eSe \in \mathbf{G}$ for all idempotents $e \in S$.
- **D** is the pseudovariety of all finite *semigroups whose idempotents are right zeros*.
- **LG** is the semidirect product $\mathbf{G} * \mathbf{D}$.

- Recall that a *pseudovariety* (of semigroups) is a class $\mathbf{V}$ of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products.
- A pseudovariety $\mathbf{V}$ is said to be *decidable* if there is an algorithm to test membership of a finite semigroup in $\mathbf{V}$.

Theorem (Rhodes'1999)

There exists a decidable pseudovariety $\mathbf{V}$ such that the semidirect product $\mathbf{Sl} * \mathbf{V}$ is not decidable, where $\mathbf{Sl}$ is the pseudovariety of all finite semilattices.

The concept of *tameness* of a pseudovariety was introduced in

J. Almeida and B. Steinberg, *On the decidability of iterated semidirect products and applications to complexity*, Proc. London Math. Soc. **80** (2000), 50–74.

with the purpose of solving the decidability problem for iterated semidirect products of pseudovarieties.That objective has not yet been fully achieved.

Theorem

If $\mathbf{V}$ and $\mathbf{W}$ are tame pseudovarieties with $\mathbf{W}$ locally finite (i.e., $\mathbf{W}$ contains the free objects in the variety it generates), then the semidirect product $\mathbf{V} * \mathbf{W}$ is decidable.

- The tameness property is parameterized by an *implicit signature* σ (a set of implicit operations on semigroups containing the multiplication) and we speak of *σ-tameness*.
- Proving the *σ-tameness* of a pseudovariety **V** involves proving two properties:
  1. that the *word problem for σ-terms* over **V** is decidable, and
  2. that **V** is *σ-reducible*.
- The most commonly used signature, known as the *canonical signature*, is $\kappa = \{ab, a^{\omega-1}\}$ consisting of the multiplication and the $(\omega - 1)$-power.

- A *κ-term* is a formal expression obtained from the letters of an alphabet A using two operations:
  - the concatenation $(x, y) \mapsto xy$
  - the $(\omega - 1)$-power $x \mapsto x^{\omega-1}$.
- A κ-term has a natural *interpretation* on each finite semigroup S:
  - the concatenation corresponds to the *semigroup multiplication*;
  - for each element $s \in S$, $s^{\omega-1}$ is the *inverse* of $s^{\omega+1}(= s^\omega s)$ in the maximal subgroup containing the unique idempotent power s^ω of s.

Theorem (Costa & Nogueira & Teixeira'2015)

The word problem for κ-terms over **LG** *is decidable.*

This talk is about κ-reducibility of semidirect products of the form $\mathbf{V} * \mathbf{D}$.
Our main result is the following.

Theorem

If $\mathbf{V}$ is a κ-reducible pseudovariety, then $\mathbf{V} * \mathbf{D}$ is also κ-reducible.

This result applies, for instance, to the pseudovarieties $\mathbf{SI}$ and $\mathbf{G}$.

Corollary (Costa & Teixeira'2004)

The pseudovariety $\mathbf{SI} * \mathbf{D} (= \mathbf{LSI})$ is κ-tame.

Corollary

The pseudovariety $\mathbf{G} * \mathbf{D} (= \mathbf{LG})$ is κ-tame.

The semidirect products of the form $\mathbf{V} * \mathbf{D}$ are among the most studied.

H. Straubing, *Finite semigroup varieties of the form* $\mathbf{V} * \mathbf{D}$, J. Pure Appl. Algebra **36** (1985), 53–94.

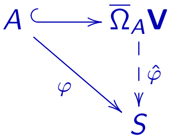
D. Thérien and A. Weiss, *Graph congruences and wreath products*, J. Pure Appl. Algebra **36** (1985), 205–215.

B. Tilson, *Categories as algebra: an essential ingredient in the theory of monoids*, J. Pure Appl. Algebra **48** (1987), 83–198.

J. Almeida and A. Azevedo, *On regular implicit operations*, Portugaliæ Mathematica **50** (1993), 35–61.

B. Steinberg, *A delay theorem for pointlikes*, Semigroup Forum **63** (2001), 281–304.

- *Pro-**V** semigroup*: a compact semigroup residually in **V**.
- $\overline{\Omega}_A\mathbf{V}$ denotes the *free pro-**V** semigroup over an alphabet A*:



where S is an arbitrary pro-**V** semigroup.

- Each pseudoword $u \in \overline{\Omega}_A\mathbf{V}$ defines an implicit operation $u_S : S^A \to S$ by $u_S(\varphi) = \hat{\varphi}(u)$.
- *Pseudoidentity*: write $S \models u = v$ if $u_S = v_S$.
- $\Omega^\sigma_A\mathbf{V}$ denotes the σ-subsemigroup of $\overline{\Omega}_A\mathbf{V}$ generated by A, whose elements are called *σ-words*.

- Let $\Sigma = \{u_i = v_i : i \in I\}$ be a finite system of equations of σ-terms over a finite set X of variables. To each variable $x \in X$ it is associated a constraint $s_x \in S$ in a given finite semigroup S.
- A **V**-*solution of the system* Σ is a mapping $\eta : X \to \overline{\Omega}_A\mathbf{S}$ together with a continuous homomorphism $\delta : \overline{\Omega}_A\mathbf{S} \to S$ such that:
  1. $\forall x \in X,\ \delta(\eta(x)) = s_x$.
  2. $\forall i \in I,\ \mathbf{V} \models \hat{\eta}(u_i) = \hat{\eta}(v_i)$.

  If $\eta(X) \subseteq \Omega^\sigma_A\mathbf{S}$, then η is called a $(\mathbf{V}, \sigma)$*-solution of* Σ.
- **V** is said *completely σ-reducible* if the existence of a **V**-solution for any such system implies the existence of a $(\mathbf{V}, \sigma)$-solution.
- **V** is said *σ-reducible* if the existence of a **V**-solution for any *system associated to a finite graph* implies the existence of a $(\mathbf{V}, \sigma)$-solution.

- A pseudovariety $\mathbf{V}$ is said to be *order-computable* when the free pro-$\mathbf{V}$ semigroup $\overline{\Omega}_A\mathbf{V}$ is finite and effectively computable.

J. Almeida, J. C. Costa and M. L. Teixeira, *Semidirect product with an order-computable pseudovariety and tameness*, Semigroup Forum **81** (2010), 26–50.

Theorem

If $\mathbf{V}$ is a κ-reducible pseudovariety and $\mathbf{W}$ is an order-computable pseudovariety, then $\mathbf{V} * \mathbf{W}$ is κ-reducible.

- Let $\mathbf{D}_n = [\![yx_1 \cdots x_n = x_1 \cdots x_n]\!]$ for each positive integer n.
- $\overline{\Omega}_A \mathbf{D}_n$ is the semigroup $\{w \in A^+ : |w| \leq n\}$ with multiplication $u \cdot v = \mathtt{t}_n(uv)$, the longest suffix of uv of length at most n.
- The $\mathbf{D}_n$ are *order-computable* pseudovarieties such that $\mathbf{D} = \bigcup_n \mathbf{D}_n$.

Corollary

If $\mathbf{V}$ is a κ-reducible pseudovariety, then $\mathbf{V} * \mathbf{D}_n$ is κ-reducible.

J. C. Costa, C. Nogueira and M. L. Teixeira, *Pointlike reducibility of pseudovarieties of the form* $\mathbf{V} * \mathbf{D}$, Int. J. Algebra Comput. **26** (2016), 203–216.

Theorem

If $\mathbf{V}$ is pointlike κ-reducible, then $\mathbf{V} * \mathbf{D}$ is also pointlike κ-reducible.

- Let $\Phi_n : A^+ \to (A^{n+1})^*$ be the function sending each word $w \in A^+$ to the sequence of factors of length $n+1$ of w in the order they occur.
- It has a unique continuous extension $\Phi_n : \overline{\Omega}_A \mathbf{S} \to (\overline{\Omega}_{A^{n+1}} \mathbf{S})^1$, which is an *n-superposition homomorphism*:
  1. $\Phi_n(w) = 1$ for every $w \in A^{\leq n}$;
  2. $\Phi_n(\pi\rho) = \Phi_n(\pi)\Phi_n(\mathtt{t}_n(\pi)\rho) = \Phi_n(\pi\mathtt{i}_n(\rho))\Phi_n(\rho)$ for every $\pi, \rho \in \overline{\Omega}_A \mathbf{S}$.

Theorem (Almeida & Azevedo'1993)

For any pseudowords $\pi, \rho \in \overline{\Omega}_A \mathbf{S}$,

$$\mathbf{V} * \mathbf{D}_n \models \pi = \rho \iff \begin{cases} \mathtt{i}_n(\pi) = \mathtt{i}_n(\rho) \\ \mathtt{t}_n(\pi) = \mathtt{t}_n(\rho) \\ \mathbf{V} \models \Phi_n(\pi) = \Phi_n(\rho) \end{cases}$$

Theorem

If $\mathbf{V}$ *is a* κ*-reducible pseudovariety, then* $\mathbf{V} * \mathbf{D}$ *is also* κ*-reducible.*

Proof (strategy)

Given a $\mathbf{V} * \mathbf{D}$*-solution* $\eta : \Gamma \to \overline{\Omega}_A\mathbf{S}$ *of the system* Σ_Γ *with respect to a continuous homomorphism* $\delta : \overline{\Omega}_A\mathbf{S} \to S$*, we have to construct a* $(\mathbf{V} * \mathbf{D}, \kappa)$*-solution* $\eta' : \Gamma \to \Omega^\kappa_A\mathbf{S}$ *of* Σ_Γ *with respect to* δ*. The proof is made in three stages.*

1. *Reduce to the case in which the solution* η *has some usefull properties.*
2. *Identify a sufficiently large integer* n *and get a* $(\mathbf{V} * \mathbf{D}_n, \kappa)$*-solution* $\eta'_n : \Gamma \to \Omega^\kappa_A\mathbf{S}$ *of* Σ_Γ *with respect to* δ*.*
3. *Transform* η'_n *into the* $(\mathbf{V} * \mathbf{D}, \kappa)$*-solution* η' *using certain (basic) transformations of the form* $a_1 \cdots a_n \mapsto a_1 \cdots a_j(a_i \cdots a_j)^\omega a_{j+1} \cdots a_n$*, which replace words of length* n *by* κ*-words.*

The *first stage* of the proof consists in reducing to the case in which the solution η verifies the properties:

- All vertices of the graph Γ are labeled by infinite pseudowords under η.
- If an edge of Γ is labeled by a finite word under η, then it is labeled by a letter.

This is obtained by easy manipulations in the graph and the solution.
The price we have to pay is that the mapping η' has to preserve some (very few and simple) properties of the mapping η. For instance, for each vertex v of Γ, $\eta'(\mathsf{v})$ and $\eta(\mathsf{v})$ must have the same prefix of a certain fixed length.

In the *second stage* of the proof one begins by identifying a positive integer n depending on the mapping η and on the semigroup S.

- Let $A^{-\infty} = A^+ \cup A^{-\mathbb{N}}$, with $A^{-\mathbb{N}}$ the set of *left-infinite words over A*.
- The set $A^{-\infty}$ is endowed with a semigroup structure by defining a product as follows:
  - A^+ is a subsemigroup of $A^{-\infty}$;
  - left-infinite words are right zeros;
  - if $w = \cdots a_{-2}a_{-1}$ is a left-infinite word and $z = b_1 b_2 \cdots b_n$ is a finite word, then wz is the left-infinite word $wz = \cdots a_{-2}a_{-1}b_1 b_2 \cdots b_n$.
- It is well-known that $\overline{\Omega}_A\mathbf{D}$ is isomorphic to the semigroup $A^{-\infty}$.

- For each vertex v of graph Γ, denote $\mathbf{d}_{\mathsf{v}} = p_{\mathbf{D}}(\eta(\mathsf{v})) \in A^{-\mathbb{N}}$ the projection of $\eta(\mathsf{v})$ into $\overline{\Omega}_A\mathbf{D}$.
- The relation $\propto$ defined by

$$\mathbf{d}_{\mathsf{v}_1} \propto \mathbf{d}_{\mathsf{v}_2} \iff \mathbf{d}_{\mathsf{v}_1} \text{ and } \mathbf{d}_{\mathsf{v}_2} \text{ are confinal left-infinite words}$$

  is an equivalence on the set $\{\mathbf{d}_{\mathsf{v}} : \mathsf{v} \in V(\Gamma)\}$.
- $n = M + Q$ for well-determined positive integers M and Q for which:
  - $\mathtt{t}_n(\eta(\mathsf{v})) = x_{\mathsf{v}} y_{\mathsf{v}} z_{\mathsf{v}}$ with $|y_{\mathsf{v}}| = M$;
  - the words y_{v} are called *borders of the solution* η;
  - if $\mathbf{d}_{\mathsf{v}_1} \propto \mathbf{d}_{\mathsf{v}_2}$ then $y_{\mathsf{v}_1} = y_{\mathsf{v}_2}$;
  - for any two distinct occurrences of borders y_{v_1} and y_{v_2} in a finite word, either these occurrences have a gap of size at least Q between them, or y_{v_1} and y_{v_2} are the same *periodic border* u^k (meaning that $\mathbf{d}_{\mathsf{v}_1} = \mathbf{d}_{\mathsf{v}_2} = u^{-\infty}$).

- As $\mathbf{V} * \mathbf{D}_n$ is a subpseudovariety of $\mathbf{V} * \mathbf{D}$, η is a $\mathbf{V} * \mathbf{D}_n$-solution.
- By a Corollary above, $\mathbf{V} * \mathbf{D}_n$ is κ-reducible since $\mathbf{V}$ also is.
- Therefore, there is a $(\mathbf{V} * \mathbf{D}_n, \kappa)$-solution $\eta'_n : \Gamma \to \Omega^\kappa_A \mathbf{S}$ of Σ_Γ with respect to δ.
- Moreover, it is possible to constrain the values $\eta'_n(\mathrm{g})$ of each $\mathrm{g} \in \Gamma$ in such a way that:
  - $\eta'_n(\mathrm{g})$ is an infinite pseudoword if $\eta(\mathrm{g})$ is infinite. In particular, $\eta'_n(\mathrm{v})$ is an infinite pseudoword for all vertices v.
  - $\eta'_n(\mathrm{g})$ and $\eta(\mathrm{g})$ have the same prefixes and the same suffixes of length at most n. In particular, for each vertex v, $\eta'_n(\mathrm{v})$ is of the form

$$\eta'_n(\mathrm{v}) = \pi_\mathrm{v} \underbrace{x_\mathrm{v} y_\mathrm{v} z_\mathrm{v}}_{\mathrm{t_v}}.$$

In *stage 3* of the proof, we associate to each word $w = a_1 \cdots a_n \in A^n$ a κ-term $\widehat{w} = a_1 \cdots a_j (a_i \cdots a_j)^\omega a_{j+1} \cdots a_n$ such that $\delta(\widehat{w}) = \delta(w)$.

Lemma

For $w = a_1 \cdots a_{n+1} \in A^{n+1}$, let $w_1 = a_1 \cdots a_n$ and $w_2 = a_2 \cdots a_{n+1}$ be the two factors of w of length n. If $\widehat{w}_1 = a_1 \cdots a_{j_1} (a_{i_1} \cdots a_{j_1})^\omega a_{j_1+1} \cdots a_n$ and $\widehat{w}_2 = a_2 \cdots a_{j_2} (a_{i_2} \cdots a_{j_2})^\omega a_{j_2+1} \cdots a_{n+1}$, then $j_1 \leq j_2$.

We then define $\psi_n : (\overline{\Omega}_{A^{n+1}}\mathbf{S})^1 \to (\overline{\Omega}_A\mathbf{S})^1$ as the only continuous monoid homomorphism which extends the mapping

$$\begin{aligned} A^{n+1} &\to \Omega_A^\kappa \mathbf{S} \\ a_1 \cdots a_{n+1} &\mapsto (a_{i_1} \cdots a_{j_1})^\omega a_{j_1+1} \cdots a_{j_2} (a_{i_2} \cdots a_{j_2})^\omega \end{aligned}$$

and let $\theta_n = \psi_n \Phi_n$.

- The function $\theta_n : \overline{\Omega}_A\mathbf{S} \to (\overline{\Omega}_A\mathbf{S})^1$ is a continuous n-superposition homomorphism since it is the composition of the continuous n-superposition homomorphism Φ_n with the continuous homomorphism ψ_n.
- A word $w = a_1 \cdots a_k$ of length $k > n$ has precisely $r = k - n + 1$ factors of length n and

$$\begin{aligned}\theta_n(w) &= \psi_n(a_1 \cdots a_{n+1}, a_2 \cdots a_{n+2}, \ldots, a_{r-1} \cdots a_k)\\ &= \psi_n(a_1 \cdots a_{n+1})\psi_n(a_2 \cdots a_{n+2}) \cdots \psi_n(a_{r-1} \cdots a_k)\\ &= (e_1^\omega f_1 e_2^\omega)(e_2^\omega f_2 e_3^\omega) \cdots (e_{r-1}^\omega f_{r-1} e_r^\omega)\\ &= e_1^\omega f_1 e_2^\omega f_2 \cdots e_{r-1}^\omega f_{r-1} e_r^\omega.\end{aligned}$$

- θ_n transforms κ-words into κ-words.

- The mapping $\eta' : \Gamma \to \Omega_A^\kappa \mathbf{S}$ is defined, for each $g \in \Gamma$, as

$$\eta'(g) = \tau_1(g)\tau_2(g)\tau_3(g),$$

where the fundamental τ_i is $\tau_2 = \theta_n \eta'_n$.

- In general, the mappings τ_1 and τ_3 serve to "give back", respectively, the prefix and the suffix of $\eta'_n(g)$ that is lost when applying θ_n.
- The exception is when e is an edge $v \xrightarrow{e} w$ such that ηe is infinite, in which case $\tau_3(v)$ and $\tau_1(e)$ must verify

$$\tau_3(v)\tau_1(e) = \theta_k(\mathtt{t}_v \mathtt{i}_e)$$

as a consequence of the graph equations. This was the most complex situation that justified most of the difficulties in stage 2.

Proposition

The mapping η' is a $(\mathbf{V} * \mathbf{D}, \kappa)$-solution of Σ_Γ with respect to δ.

What about *complete reducibility* of pseudovarieties of the form $\mathbf{V} * \mathbf{D}$?
Does the following statement hold?

Conjecture

If $\mathbf{V}$ is a completely κ-reducible pseudovariety, then $\mathbf{V} * \mathbf{D}$ is also completely κ-reducible.

This appears to be a much more challenging problem.