Decidability of complexity via pointlike sets: A premodern, hyper-elementary and explicitly constructive approach satisfying MC

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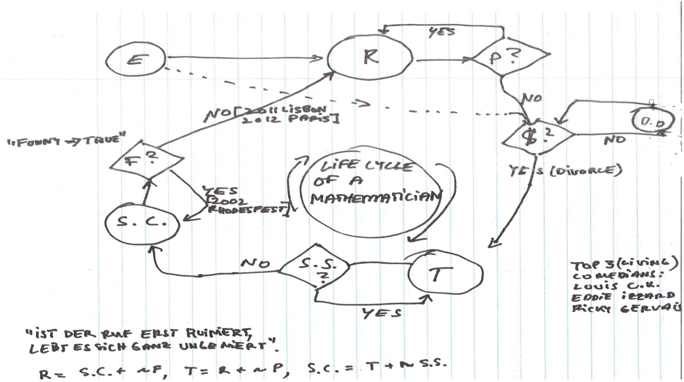
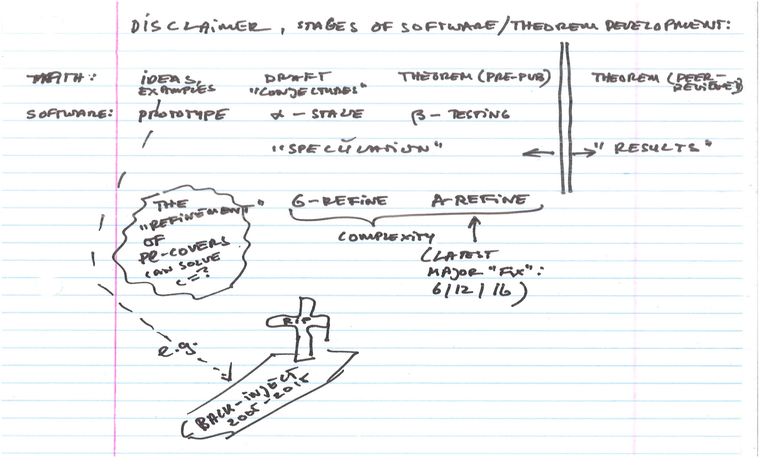
Personal Note:
My aim in this talk is to demonstrate the promise of the "cover-refinement" approach to the complexity problem. I have been working on complexity for over 40 years, and a solution using the "cover-refinement" approach seems to be within reach...
diagram: life cycle of a mathematician



yet it has also become clear to me, that without help I will not be able to publish these results. I am therefore extending an open invitation to collaborate (and to co-author) to anyone interested in this approach.
diagram: disclaimer, stages of software or theorem development



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"Meakin - Condition (MC)" : no more "lower bounds" papers...

Some definitions:

$\mathbf{A}$ = (pseudovariety of finite) aperiodic semigroups,

$\mathbf{G}$ = (pseudovariety of finite) groups,

$\mathbf{V_n} = (\mathbf{A} * \mathbf{G})^{\mathbf{n}} * \mathbf{A}$,

$P(S)$ = the power set of S,

$[S] = \{\{s\} | s \in S\}$,

$\bigcup : P^2(S) \to P(S)$ is the union - map,

$Z(g)$ = the cyclic group generated by (a group element) g,

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$Z(g)$ = the cyclic group generated by (a group element) g,

g is a group element iff $g^{\omega+1} = g$,

$c(S)$ = minimal n such that $S \in \mathbf{V_n}$ = complexity of S,

$Pl_{\mathbf{V}}(S) = \{X \in S|$ for all relations $R : S \to V$ with $V \in \mathbf{V}$ there exists a $v \in V$ such that $X \subseteq R^{-1}(v)\}$.

$Pl_{\mathbf{V}}(S))$ is called the $\mathbf{V}$ - pointlike sets of S.

Then an alternate characterization of $c(S)$ is

$$c(S) = \text{the minimal } n \text{ such that } Pl_{\mathbf{V_n}}(S) = [S].$$

Given a relation $R : S \to T$, define

$$C(R) = \{R^{-1}(t) | t \in T\} \text{ , closed under products and subsets.}$$

$C(R)$ is called the cover-semigroup presented by R ; we also say R computes $C(R)$.

If $T \in \mathbf{V}$ we say $C(R)$ is $\mathbf{V}$ - presentable or $C(R)$ is $\mathbf{V}$ - presented (depending on if we assert the existence of an R, or actually exhibit R).

It is well known that $Pl_{\mathbf{V}}(S)$ is $\mathbf{V}$ - presentable.

Our goal (the "cover-refinement approach to complexity") is to determine (recursively) $Pl_{\mathbf{W}*\mathbf{V}}(S)$ [for all S], given complete information about $Pl_{\mathbf{V}}(S)$ [for all S].

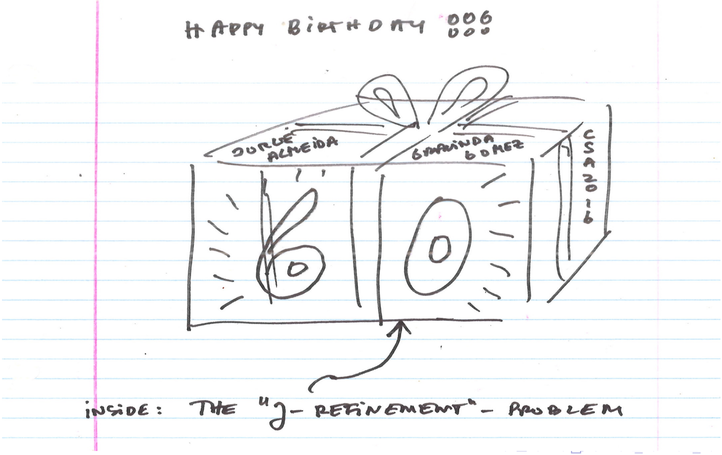
Ideally, we would accomplish this constructively, i.e. determine a $\mathbf{W} * \mathbf{V}$ - presentation for $Pl_{\mathbf{W}*\mathbf{V}}(S)$ (based on $\mathbf{V}$ - presentations for $Pl_{\mathbf{V}}(S)$ [for various S]).

This ambitious program (the "cover-refinement approach to complexity") has mostly been worked out for the aperiodic case

$\mathbf{W} = \mathbf{A}$,

while the group case $\mathbf{W} = \mathbf{G}$ is still under investigation.

You probably already know all about $PI_{\mathbf{J}}$.
But here is a nice research problem:
(" J - refinement"):
can you determine $PI_{\mathbf{J} * \mathbf{V}}$, given everything you want to know about $PI_{\mathbf{V}}$?



Part I: the aperiodic case $\mathbf{W} = \mathbf{A}$:

Define $C_{\mathbf{A}}(S) =$ the smallest cover-semigroup C such that $[S] \leq C \leq P(S)$ and C is closed under

(*) if $g \in C$ is a group element with $Z(g) \in Pl_{\mathbf{V}}(C)$, then $\bigcup Z(g) \in C$.

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(1) Review of $Pl_{\mathbf{A}}$:

The critical Pl-generation equation

(*) if $g \in C$ is a group element with $Z(g) \in Pl_{\mathbf{V}}(C)$, then $\bigcup Z(g) \in C$.

becomes for $\mathbf{V} = \mathbf{1}$ "any group can be unioned".

So, form the "product expansion" of $C_{\mathbf{A}}$, but modify the action of regular elements $X \in C_{\mathbf{A}}$ by "blowing up" first.

diagram: the product expansion

To "blow up X" take the (right) group of X, $G(X)$, and multiply X by $\bigcup G(X)$ (you may have to do this repeatedly...).

Note that in the product expansion the action at Null classes is already aperiodic, so we never have to worry about "blowing up" null elements; also in order to maintain J - strings all that is required of the "blow up" is that it goes down in the J- order.

Then the only regular J-classes that are really used in the modified product expansion are aperiodic, hence the modified product expansion is aperiodic.

(2) The recursion $Pl_{\mathbf{A}*\mathbf{V}}(-)$ from $Pl_{\mathbf{V}}(-)$:

Basic **A** -construction Lemma:

Let $g \in Pl_{\mathbf{A}*\mathbf{V}}(S)$ be a group element, and let $Z(g) \in Pl_{\mathbf{V}}(Pl_{\mathbf{A}*\mathbf{V}}(S))$, then $\bigcup Z(g) \in Pl_{\mathbf{A}*\mathbf{V}}(S)$.

(Note that we need information of $Pl_{\mathbf{V}}$ at $Pl_{\mathbf{A}*\mathbf{V}}(S)$ ... !).

To prove this, you really need to establish two things:

(a) that your construction operator is "functorial", i.e. "lifts" and "pushes",

and

(b) that your construction operator, applied to an $A \circ V \in \mathbf{A} * \mathbf{V}$ gives nothing but singletons.

to see (b), if you take a group element $g \in A \circ V$ with $Z(g) \in Pl_{\mathbf{V}}(A \circ V)$, then the projection of $Z(g)$ unto V will be an idempotent e (since $Z(g)$ is **V** - pointlike !). But then the inverse image of e will be aperiodic, and so $Z(g) = \{g^{\omega}\}$ is also just a point.

The Basic **A** -construction Lemma then insures that

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For the opposite direction we adapt our proof (for $\mathbf{V} = \mathbf{1}$) in "Product expansions" (JPAA 101 (1995) pp 157-170, reviewed above) to the general case.

Preliminary note:

In any attempt to refine a cover by some additional "computational resource", there are two different "models" to try:

Model (1) ("liftable pairs") : Use pairs (X, Y) with $X \subseteq Y$ and $X \in Pl_{\mathbf{A}*\mathbf{V}}(S)$ and $Y \in Pl_{\mathbf{V}}(S)$.

This only works well if the pairs are "liftable", and I have never been able to make this approach work...

So the determination of "liftable pairs" is an open research problem.

The definition of "liftable pairs" is as follows:

$(X, Y) \in LPl_{\mathbf{W},\mathbf{V}}(S)$ iff for all relations $R : S \to W \circ V$ with $W \circ V \in \mathbf{W} * \mathbf{V}$ there exists a $w \in W \circ V$ such that $X \subseteq R^{-1}(w)\}$ and $Y \subseteq (R \circ proj_{\mathbf{V}})^{-1}(proj_{\mathbf{V}}(w))$.

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Model (2) ("$\mathbf{V}$-bounding the set X"): Use pairs (X, Y) with $X \in Y$, where $X \in Pl_{\mathbf{A}*\mathbf{V}}(S)$ and $Y \in Pl_{\mathbf{V}}(Pl_{\mathbf{A}*\mathbf{V}}(S))$.
we have only been able to get model (2) to work....

Theorem 2: $Pl_{\mathbf{A}*\mathbf{V}}(S) \leq C_{\mathbf{A}}(S)$ (constructively)

The indicated modification goes as follows:

First, pick any relation $R : C_{\mathbf{A}}(S) \to V$ that computes **V** - pointlike sets of $C_{\mathbf{A}}(S)$.

It is important to note that we do not place any additional requirements on R; all it has to do is compute the right pointlike sets.

Call a regular J -class of $C_{\mathbf{A}}(S)$ "**V** - stable" iff $R^{-1}(e') \cap G(E)$ is a singleton. Null J - classes are automatically stable

It is critically important that we can show that

(1) every regular stable J - class J has a counterpart J_V in V that "computes the group coordinate of J perfectly", i.e. let $e \in J$ and $e_V \in J_V$ be idempotents that are R - related, then $R^{-1}(e_V) \cap G(e)$ is a singleton,

(2) for every regular J -class of $C_{\mathbf{A}}(S)$ we can descend from it in R - order to get to a stable one. This descent makes sets larger ("blow -up"). This is so because $R^{-1}(e_V) \cap G(e)$ in general is a V - pointlike set.

Then in the modified version of the product expansion of $C_{\mathbf{A}}(S)$, the only regular J- classes that are used are stable.

We then (roughly) construct a semigroup computing the desired pointlikes as follows:

In the right hand coordinate use the Rhodes expansion $\widehat{V}$ of V, in the left hand coordinate use the modified product expansion of $C_{\mathbf{A}}(S)$ (use only stable J - classes and strip out the group coordinate), together with base point B_0 and a pointer into the Rhodes expansion to get the multiplier m.

Note that we actually construct a presentation of $C_{\mathbf{A}}(S)$ from the **V** - presentation $R : C_{\mathbf{A}}(S) \to V$ that computes **V** - pointlike sets of $C_{\mathbf{A}}(S)$.

Bonus CSA 2016 Corollary:
if $\mathbf{V}$ is closed under $\mathbf{A}*$, then $Pl_{\mathbf{V}}$ has a nice representation as the minimal C closed under $\bigcup G$ for G from some characterizing set of subgroups of $P(S)$.

Also, knowing HOW $Pl_{\mathbf{V}}$ can be computed, can help in trying to figure out how to compute $Pl_{\mathbf{G}*\mathbf{V}}$. We can "invest" an $\mathbf{A}$ for free (as far as complexity is concerned)and thereby know something about HOW $\mathbf{V}$ goes about its business.