Crystal monoids and crystal bases

Robert D. Gray[1]
University of East Anglia

CSA International Conference on
Semigroups and Automata 2016

Celebrating the 60th birthdays of
Jorge Almeida and Gracinda Gomes

Funded by: IF/01622/2013/CP1161/CT0001

[1](joint work with A. J. Cain and A. Malheiro)

Gracinda e Jorge :-)

The importance of conferences

- July 2011: *Groups and Semigroups: Interactions and Computations, Lisbon*.

Efim Zelmanov asked: Can finite state automata be used to compute efficiently with Plactic monoids?

This led A. J. Cain, A. Malheiro and me to get interested in Plactic monoids and algebras.

The importance of conferences

- July 2011: *Groups and Semigroups: Interactions and Computations, Lisbon*.

Efim Zelmanov asked: Can finite state automata be used to compute efficiently with Plactic monoids?

This led A. J. Cain, A. Malheiro and me to get interested in Plactic monoids and algebras.

- June 2013: *Geometric, Combinatorial & Dynamics Aspects of Semigroups and Groups, On the occasion of the 60th birthday of Stuart Margolis Bar-Ilan, Israel*.

Anne Schilling pointed out connections with crystal basis theory (in the sense of Kashiwara (1990)).

Plactic monoid

Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.

I want to give three different ways of defining a certain equivalence relation $\sim$ on the free monoid $\mathcal{A}_n^*$ of all words:

1. Presentation (Knuth relations)
2. Tableaux (Schensted insertion algorithm)
3. Crystal bases (in the sense of Kashiwara)

We call $\sim$ the Plactic congruence and the resulting quotient monoid $\mathrm{Pl}(A_n) = \mathcal{A}_n^*/\sim$ is called the Plactic monoid (of rank n).

The Plactic monoid

- Has origins in work of Schensted (1961) and Knuth (1970) concerned with combinatorial problems on Young tableaux.
- Later studied in depth by Lascoux and Shützenberger (1981).

Due to close relations to Young tableaux, has become a tool in several aspects of representation theory and algebraic combinatorics.

Applications of the Plactic monoid

- proof of Littlewood–Richardson rule for Schur functions (an important result in the theory of symmetric functions)
  - appendix of J. A. Green's "Polynomial representations of GL_n".
- combinatorial description of Kostka–Foulkes polynomials, which arise as entries of the character table of the finite linear groups.

M. P. Schützenberger 'Pour le monoïde plaxique' (1997)

Argues that the Plactic monoid ought to be considered as "one of the most fundamental monoids in algebra".

Plactic monoid via Knuth relations

Definition

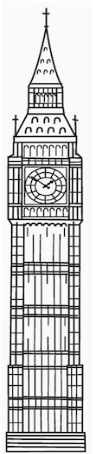
Let $\mathcal{A}_n$ be the finite ordered alphabet $\{1 < 2 < \ldots < n\}$.
Let $\mathcal{R}$ be the set of defining relations:

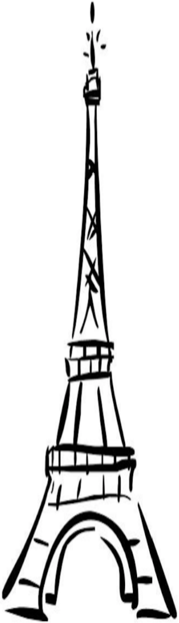
$$zxy = xzy \quad \text{and} \quad yzx = yxz \qquad x < y < z,$$
$$xyx = xxy \quad \text{and} \quad xyy = yxy \qquad x < y.$$

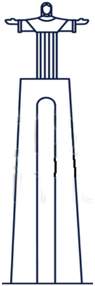
The Plactic monoid $\mathrm{Pl}(A_n)$ is defined by the presentation $\langle \mathcal{A}_n | \mathcal{R} \rangle$.
$\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ where $\sim$ is the smallest congruence on the free monoid $\mathcal{A}_n^*$ containing $\mathcal{R}$.

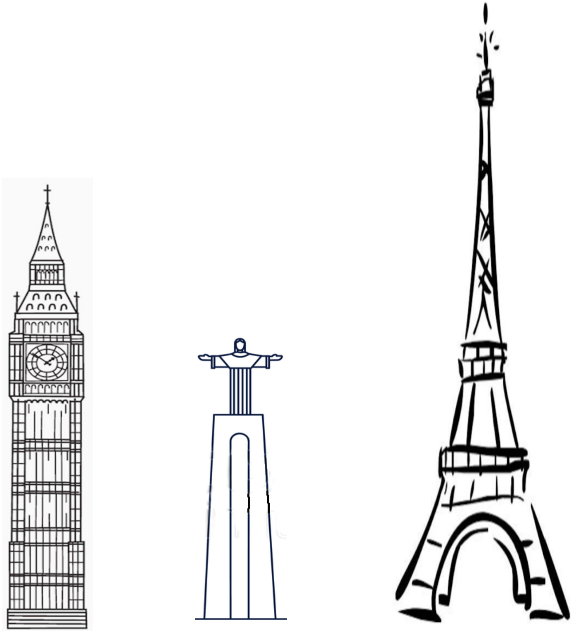
$$\text{e.g. } 212313 \sim 212133$$

- This is the most efficient way to define the Plactic congruence $\sim$.
- The relations in this presentation are called the Knuth relations.









A (semi-standard) tableau

1	1	1	2	2	4	4
2	2	3	3			
4	5	5	6			
6	8					

Properties

- Is a filling of the Young diagram with symbols from $\mathcal{A}_n$.
- Rows read left-to-right are non-decreasing.
- Columns read down are strictly increasing.
- Longer rows are above shorter rows.

Schensted column insertion algorithm

- Associates to each word $w \in \mathcal{A}_n^*$ a tableau $P(w)$.
- The algorithm which produces $P(w)$ is recursive.

Input: Any letter $x \in \mathcal{A}_n$ and a tableau T.

Output: A new tableau denoted $x \to T$.

The idea: Suppose $T = C_1 C_2 \ldots C_r$ where C_i are the columns of T.

- We try to insert the box $\boxed{x}$ under the column C_1 if we can.
- If this fails, the box $\boxed{x}$ will be put into column C_1 higher up and will "bump out" to the right a box $\boxed{y}$ where y is the minimal letter in C_1 such that $x \leq y$.
- We then take the bumped out box $\boxed{y}$ and try and insert it under the column C_2, and so on...

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\boxed{2},$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array}, \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array},$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\boxed{2}, \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array},$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$
\begin{array}{|c|}\hline 2 \\ \hline \end{array}\,,\quad
\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}\,,\quad
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,,\quad
\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}\,,
$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array}\,,\quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\,,$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}\,,\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3\\\cline{1-2} 4\\\cline{1-1}\end{array} = P(w).$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{|c|}\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3\\\cline{1-2} 4\\\cline{1-1}\end{array} = P(w).$$

Observation: $231 = 213$ is a Knuth relation and $P(231) = P(213)$

$$\begin{array}{|c|}\hline 2\\\hline\end{array},\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = P(231), \qquad \begin{array}{|c|}\hline 2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = P(213).$$

Schensted's column insertion algorithm

Example

$\mathcal{A}_4 = \{1 < 2 < 3 < 4\}$ if $w = 232143$ then $P(w)$ is obtained as:

$$\begin{array}{c}2\end{array},\quad \begin{array}{c}2\\3\end{array},\quad \begin{array}{cc}2&2\\3&\end{array},\quad \begin{array}{ccc}1&2&2\\3&&\end{array},\quad \begin{array}{ccc}1&2&2\\3&&\\4&&\end{array},\quad \begin{array}{ccc}1&2&2\\3&3&\\4&&\end{array} = P(w).$$

Observation: $231 = 213$ is a Knuth relation and $P(231) = P(213)$

$$\begin{array}{c}2\end{array},\quad \begin{array}{c}2\\3\end{array},\quad \begin{array}{cc}1&2\\3&\end{array} = P(231), \qquad \begin{array}{c}2\end{array},\quad \begin{array}{cc}1&2\end{array},\quad \begin{array}{cc}1&2\\3&\end{array} = P(213).$$

Theorem (Lascoux and Shützenberger (1981))

Define a relation $\sim$ on $\mathcal{A}_n^*$ by $u \sim w \Leftrightarrow P(u) = P(w)$. Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^*/\sim$ is the Plactic monoid.

The Plactic monoid via tableaux

$\mathrm{w}(T) =$ the word obtained by reading the columns of a tableau T from right to left and top to bottom (Japanese reading).

Example: If $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ then $\mathrm{w}(T) = 415123$.

The Plactic monoid via tableaux

$\mathrm{w}(T) =$ the word obtained by reading the columns of a tableau T from right to left and top to bottom (Japanese reading).

Example: If $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ then $\mathrm{w}(T) = 415123$.

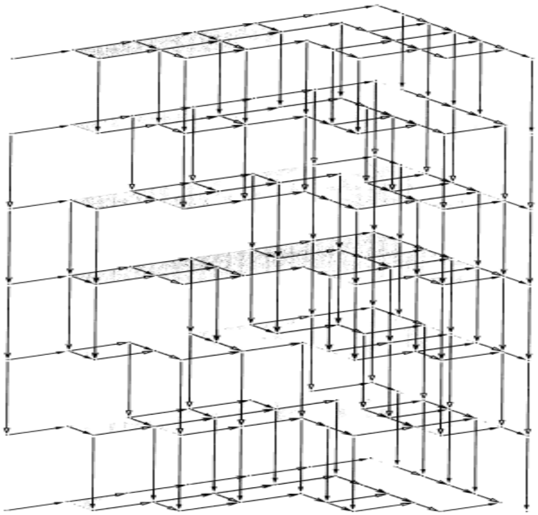
Theorem (Lascoux and Shützenberger (1981))

The set of word readings of tableaux gives a transversal (a set of normal forms) of the $\sim$-classes of the Plactic monoid.

Conclusion: The Plactic monoid is the monoid of tableaux:

- **Elements** The set of all tableaux over $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$.
- **Products** Computed using Schensted insertion.

Crystals



[2]

[2]Fig 8.4 from Hong and Kang's book *An introduction to quantum groups and crystal bases.*

Crystal graphs

(following Kashiwara and Nakashima (1994))

Idea: Define a directed labelled digraph Γ_{A_n} with the properties:

- Vertex set = $\mathcal{A}_n^*$
- Each directed edge is labelled by a symbol from the label set $I = \{1, 2, \ldots, n-1\}$.
- For each vertex $u \in \mathcal{A}_n^*$ every $i \in I$ there is at most one directed edge labelled by i leaving u, and at most one entering u

$$u \xrightarrow{\quad i \quad} v \,, \qquad\qquad w \xrightarrow{\quad i \quad} u$$

- If $u \xrightarrow{\quad i \quad} v$ then $|u| = |v|$, so words in the same component have the same length as each other. In particular, connected components are all finite.

Building the crystal graph $\Gamma_{\mathcal{A}_n}$

$$\mathcal{A}_n = \{1 < 2 < \ldots < n\}$$

We begin by specifying structure on the words of length one

$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

This is known as a Crystal basis.

Kashiwara operators

For each $i \in \{1, \ldots, n-1\}$ we define partial maps e_i and f_i on the letters $\mathcal{A}_n$ called the Kashiwara crystal graph operators. For each edge

$$a \xrightarrow{i} b \quad ,$$

we define $f_i(a) = b$ and $e_i(b) = a$.

Kashiwara operators on words

Let $u \in \mathcal{A}_n^*$ and $i \in I$.

Question: Are either / both of the following edges in Γ_{A_n}?

$$u \xrightarrow{i} f_i(u), \qquad e_i(u) \xrightarrow{i} u$$

Algorithm:

- Under each letter a of w write
  - $+$ if $f_i(a)$ is defined, and
  - $-$ if $e_i(a)$ is defined.
- Take this string of $-$'s and $+$'s and delete all adjacent $+-$.
- The resulting string is then of the form $-^q+^r$.
- $f_i(w)$: obtained by applying f_i to the letter a above the leftmost remaining $+$, if it exists, otherwise is undefined.
- $e_i(w)$: obtained by applying e_i to the letter a above the rightmost remaining $-$, if it exists, otherwise is undefined.

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} f_i(a) \quad , \qquad e_i(b) \xrightarrow{i} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$3 \quad 3 \quad 2 \quad 1 \quad 2 \quad 3 \quad 1 \quad 3 \quad 2 \quad 3 \quad 2$$

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} f_i(a) \quad , \qquad e_i(b) \xrightarrow{i} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{matrix} 3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\ & & + & & + & & & & + & & + \end{matrix}$$

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} f_i(a) \quad , \qquad e_i(b) \xrightarrow{i} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{array}{ccccccccccc} 3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\ - & - & + & & + & - & & - & + & - & + \end{array}$$

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

$$a \xrightarrow{\;i\;} f_i(a) \,, \qquad\qquad e_i(b) \xrightarrow{\;i\;} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\
- & - & + & & + & - & & - & + & - & + \\
- & - & \cancel{+} & & \cancel{+} & \cancel{-} & & \cancel{-} & \cancel{+} & \cancel{-} & +
\end{array}$$

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$a \xrightarrow{i} f_i(a) \quad , \qquad e_i(b) \xrightarrow{i} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

3	3	2	1	2	3	1	3	2	3	2
−	−	+		+	−		−	+	−	+
−	−	~~+~~		~~+~~	~~−~~		~~−~~	~~+~~	~~−~~	+
−	−									+

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

$$a \xrightarrow{\;i\;} f_i(a) \quad , \qquad\qquad e_i(b) \xrightarrow{\;i\;} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\
- & - & + & & + & - & & - & + & - & + \\
- & - & \cancel{+} & & \cancel{+} & \cancel{-} & & \cancel{-} & \cancel{+} & \cancel{-} & + \\
- & \color{blue}{-} & & & & & & & & & \color{blue}{+} \\
 & & & & & & & & & & \\
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & \color{blue}{3} = f_2(u)
\end{array}$$

Example: Computation of $e_i(u)$ and $f_i(u)$

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

$$a \xrightarrow{\;i\;} f_i(a) \quad , \qquad e_i(b) \xrightarrow{\;i\;} b$$

Example

Let $u = 33212313232$ and let $i = 2 \in I = \{1, 2\}$.

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 \\
- & - & + & & + & - & & - & + & - & + \\
- & - & \cancel{+} & & \cancel{+} & \cancel{-} & & \cancel{-} & \cancel{+} & \cancel{-} & + \\
- & \color{blue}{-} & & & & & & & & & \color{blue}{+}
\end{array}$$

$$\begin{array}{ccccccccccc}
3 & 3 & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & \color{blue}{3} = f_2(u) \\
3 & \color{blue}{2} & 2 & 1 & 2 & 3 & 1 & 3 & 2 & 3 & 2 = e_2(u)
\end{array}$$

The crystal graph Γ_{A_n}

Definition

The crystal graph Γ_{A_n} is the directed labelled graph with:

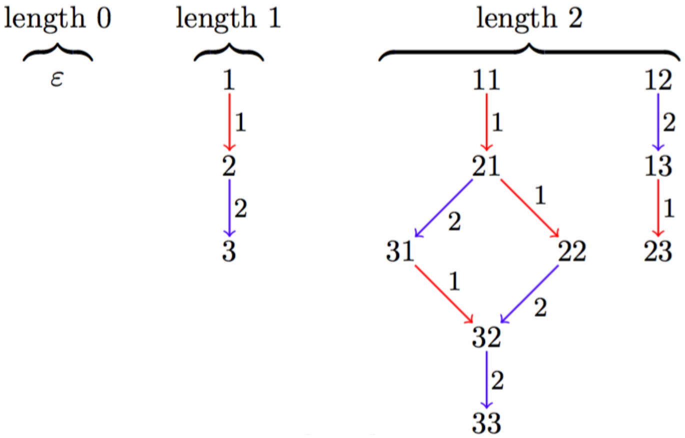
- Vertex set: $\mathcal{A}_n^*$
- Directed labelled edges: for $u \in \mathcal{A}_n^*$

$$u \xrightarrow{\;i\;} f_i(u) \quad , \qquad e_i(u) \xrightarrow{\;i\;} u$$

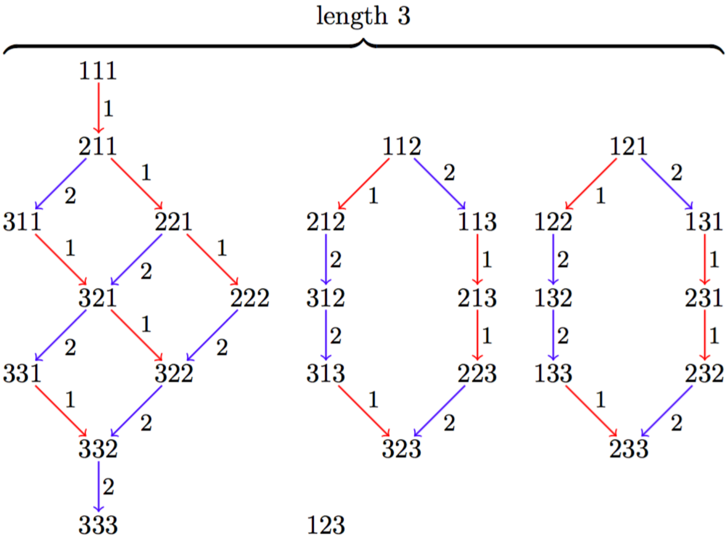
Notes

- When defined $e_i(f_i(u)) = u$ and $f_i(e_i(u)) = u$.
- It follows from the definition that (when defined) we have $e_i(u) = u' e_i(a) u''$ for some decomposition $u \equiv u' a u''$ where a is a single letter.

Part of the crystal graph for $\mathcal{A}_3 = \{1 < 2 < 3\}$



Part of the crystal graph for $\mathcal{A}_3 = \{1 < 2 < 3\}$



Plactic monoid via crystals

Definition: Two connected components $B(w)$ and $B(w')$ of Γ_{A_n} are isomorphic if there is a label-preserving digraph isomorphism $f : B(w) \to B(w')$.

Fact: In Γ_{A_n} if $B(w) \cong B(w')$ then there is a unique isomorphism $f : B(w) \to B(w')$.

Plactic monoid via crystals

Definition: Two connected components $B(w)$ and $B(w')$ of Γ_{A_n} are isomorphic if there is a label-preserving digraph isomorphism $f : B(w) \to B(w')$.

Fact: In Γ_{A_n} if $B(w) \cong B(w')$ then there is a unique isomorphism $f : B(w) \to B(w')$.

Theorem (Kashiwara and Nakashima (1994))

Let Γ_{A_n} be the crystal graph with crystal basis

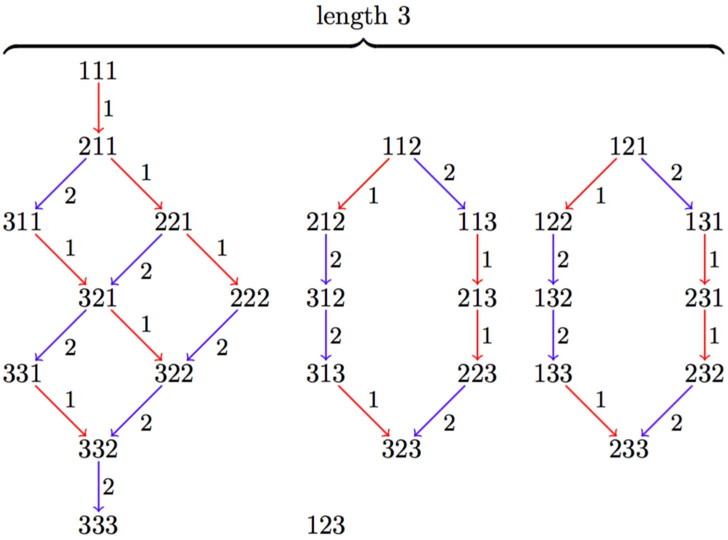
$$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$$

Define a relation $\sim$ on $\mathcal{A}_n^*$ by

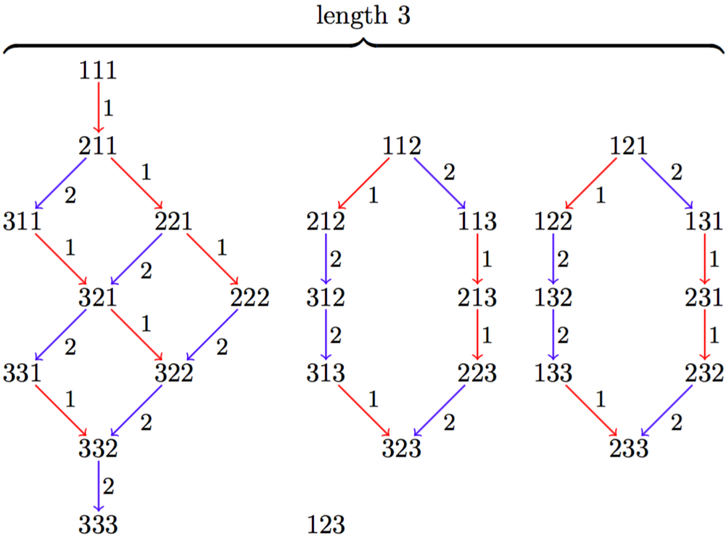
$$u \sim w \Leftrightarrow \exists \text{ an isomorphism } f : B(u) \to B(w) \text{ with } f(u) = w.$$

Then $\sim$ is the Plactic congruence and $\mathrm{Pl}(A_n) = \mathcal{A}_n^* / \sim$ is the Plactic monoid.

Knuth relations via crystal isomorphisms

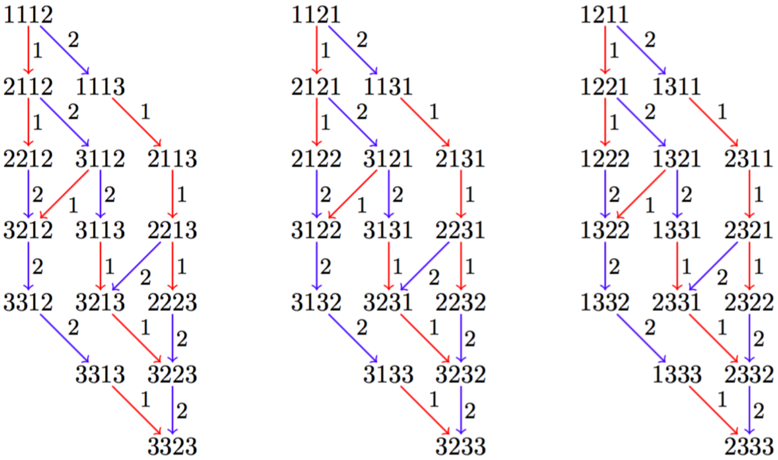


Knuth relations via crystal isomorphisms[3]



[3](**Confession:** I lied a bit. Actually, crystal isomorphisms must also preserve "weight". For $\mathrm{Pl}(A_n)$ weight preserving means "content preserving".)

Three isomorphic components for $\mathcal{A}_3 = \{1 < 2 < 3\}$.



2113, 2131, and 2311 all represent the same element.

Where do crystals come from?

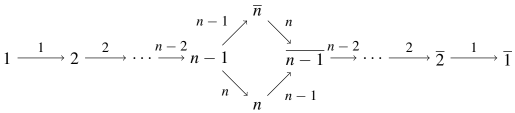
J. Hong, S.-J. Kang,
Introduction to Quantum Groups and Crystal Bases.
Stud. Math., vol. 42, Amer. Math. Soc., Providence, RI, 2002.

- Take a "nice" Lie algebra $\mathfrak{g}$. Nice means symmetrizable Kac-Moody Lie algebra e.g. a finite-dimensional semisimple Lie algebra.
- From $\mathfrak{g}$ construct its universal enveloping algebra $U(\mathfrak{g})$ which is an associative algebra.
- Drinfeld and Jimbo (1985): defined q-analogues $U_q(\mathfrak{g})$, quantum deformations, with parameter q
  - $q = 1$: $U_q(\mathfrak{g})$ coincides with $U(\mathfrak{g})$
  - $q = 0$: is called crystallisation (Kashiwara (1990)).

Where do crystals come from?

- Crystal bases are bases of $U_q(\mathfrak{g})$-modules at $q = 0$ that satisfy certain axioms.
  - Kashiwara (1991): proves existence and uniqueness of crystal bases of finite dimensional representations of $U_q(\mathfrak{g})$.
- Every crystal basis has the structure of a coloured digraph (called a crystal graph). The structure of these coloured digraphs has been explicitly determined for certain semisimple Lie algebras (special linear, special orthogonal, symplectic, some exceptional types).
- The crystal constructed from the crystal basis using Kashiwara operators is then a useful combinatorial tool for studying representations of $U_q(\mathfrak{g})$.
  - e.g. For decomposing tensor products of $U_q(\mathfrak{g})$-modules.

Crystal bases and crystal monoids

Lie algebra type	Crystal basis	Monoid
A_n: $\mathfrak{sl}_{n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \xrightarrow{n-1} n$	$\mathrm{Pl}(A_n)$
B_n: $\mathfrak{so}_{2n+1}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(B_n)$
C_n: $\mathfrak{sp}_{2n}$	$1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-1} n \xrightarrow{n} \overline{n} \xrightarrow{n-1} \cdots \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(C_n)$
D_n: $\mathfrak{so}_{2n}$		$\mathrm{Pl}(D_n)$
G_2	$1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{1} 0 \xrightarrow{1} \overline{3} \xrightarrow{2} \overline{2} \xrightarrow{1} \overline{1}$	$\mathrm{Pl}(G_2)$

Crystal monoids in general

Combinatorial crystals

- Crystal basis = finite labelled directed graph, vertex set X, label set I, satisfying certain axioms so that Kashiwara operators e_i, f_i $(i \in I)$ are well defined.
- A weight function $\mathrm{wt} : X^* \to P$ where P is the weight monoid.
- Construct a (weighted) crystal graph Γ_X from this data
  - Vertex set: X^*
  - Directed labelled edges: determined by e_i, f_i

Definition (Crystal monoid)

Let Γ_X be a crystal graph. Define $\approx$ on X^* where $u \approx v$ if there is a (weight preserving) isomorphism $\theta : B(u) \to B(v)$ with $\theta(u) = v$. Then $\approx$ is a congruence on X^* and $X^*/\approx$ is called the crystal monoid of Γ_X.

Known results and our interest

Known results on crystals A_n, B_n, C_n, D_n, or G_2 and their monoids:

1. Crystal bases - combinatorial description Kashiwara and Nakashima (1994).
2. Tableaux theory and Schensted-type insertion - Kashiwara and Nakashima (1994), Lecouvey (2002, 2003, 2007).
3. Finite presentations via Knuth-type relations - Lecouvey (2002, 2003, 2007).

Known results and our interest

Known results on crystals A_n, B_n, C_n, D_n, or G_2 and their monoids:

1. Crystal bases - combinatorial description Kashiwara and Nakashima (1994).
2. Tableaux theory and Schensted-type insertion - Kashiwara and Nakashima (1994), Lecouvey (2002, 2003, 2007).
3. Finite presentations via Knuth-type relations - Lecouvey (2002, 2003, 2007).

General question: To what extent can tools from theoretical computer science and formal language theory such as

- Finite complete (Noetherian and confluent) rewriting systems
- Finite state automata

be used to compute efficiently with crystals and crystal monoids?

Our results so far: give positive answers for all of the above types.

Automatic structures

Automatic groups and monoids

Defining property: $\exists$ a regular language $L \subseteq A^*$ such that every element has at least one representative in L, and $\forall a \in A \cup \{\epsilon\}$, there is a finite automaton recognising pairs from L that differ by multiplication by a.

- Automatic groups
  - Capture a large class of groups with easily solvable word problem
  - Examples: finite groups, free groups, free abelian groups, various small cancellation groups, Artin groups of finite and large type, Braid groups, hyperbolic groups.
- Automatic semigroups and monoids
  - Classes of monoids that have been shown to be automatic include divisibility monoids and singular Artin monoids of finite type.

Proposition (Campbell, Robertson, Ruškuc & Thomas (2001))

Automatic monoids have word problem solvable in quadratic time.

Automatic structures for crystal monoids

Theorem (Cain, RG, Malheiro (2015))

The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ are all automatic. In particular each of these monoids has word problem that is solvable in quadratic time.

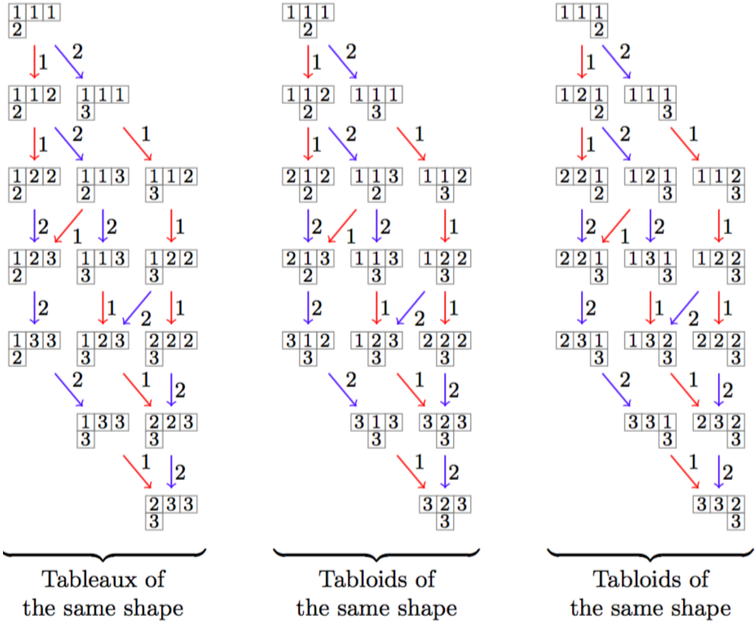
Automatic structures for crystal monoids

Theorem (Cain, RG, Malheiro (2015))

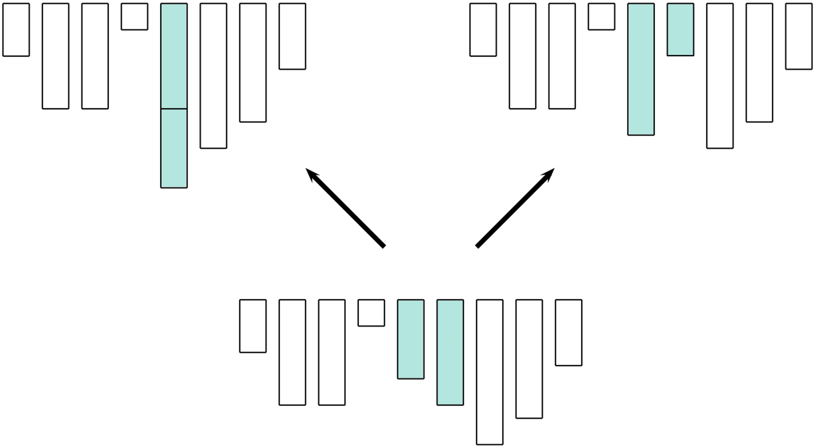
The monoids $\mathrm{Pl}(A_n)$, $\mathrm{Pl}(B_n)$, $\mathrm{Pl}(C_n)$, $\mathrm{Pl}(D_n)$, and $\mathrm{Pl}(G_2)$ are all automatic. In particular each of these monoids has word problem that is solvable in quadratic time.

- In each case there is a tableau theory, and we use a larger generating set Σ of admissible columns.
- For each $X \in \{A_n, B_n, C_n, D_n, G_2\}$ we construct a finite complete rewriting system (Σ, T) that presents $\mathrm{Pl}(X)$.
- A tabloid is a sequence of admissible columns. The rewriting system rewrites tabloids $\rightsquigarrow$ tableaux.
- Regular language of representatives for the automatic structure is the language of irreducible words of (Σ, T).
- Crystal bases theory $\rightsquigarrow_{\text{reduces problem to}} \rightsquigarrow$ highest-weight words.

Kashiwara operators preserve shape



Rewriting tabloids



- Multiplying two adjacent admissible columns of a tabloid brings us one step closer to being a tableau.

Crystal-theoretic consequences

Corollary (Cain, RG, Malheiro (2015))

For the crystal graphs of types A_n, B_n, C_n, D_n, or G_2, there is a quadratic-time algorithm that takes as input two vertices and decides whether they lie in the same position in isomorphic components.

Corollary (Cain, RG, Malheiro (2015))

For the crystal graphs of types A_n, B_n, C_n, D_n, or G_2, there is a quadratic-time algorithm that takes as input two vertices and decides whether they lie in isomorphic components.

Ongoing and future work

- Are there any further consequences to be drawn from our results
  - For crystals? For Lie theory?
- Implications for the Plactic algebras of Littelmann (1996)?

Ongoing and future work

- Are there any further consequences to be drawn from our results
  - For crystals? For Lie theory?
- Implications for the Plactic algebras of Littelmann (1996)?

We are developing further the general theory of crystal monoids.

- Examples of crystal monoids (with weight monoid $\mathbb{Z}^m$)
  - free monoids, free commutative monoids, the bicyclic monoid, the Thompson monoid (?), ...
- Squier graph / crystal graph duality.
- Finite presentations / complete rewriting systems / automatic structures?
- What can we say about complexity of the word problem?
- When do we have a tableaux theory? Highest weight words?