INTERNATIONAL CONFERENCE ON SEMIGROUPS AND AUTOMATA

Celebrating the 60th birthday of Gracinda Gomes and Jorge Almeida

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PARTIAL ACTIONS AND SUBSHIFTS OF INFINITE TYPE

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(no subject)

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Fri, Jan 13, 2006 at 7:23 PM

Caro Prof Exel

Alguns de nos aqui no Centro de Algebra na Univ Lisboa estamos interessados no seu trabalho,
gostaria de saber se estaria interessado em nos visitar e dar aqui um curso

se assim fosse nos tentariamos encontrar apoio financeiro - neste momento nao sei se tal e possivel mas ha onde tentar

nao sei se a sua universidade esta aberta no final deste mes e se o Professor ai vai estar
vou passar por Florianopolis durante um curta visita nos dias 29 e 30 de jan e poderia tentar falar consigo

escrevo-lhe na qualidade de Coordenadora do CAUL

os meus cumprimentos

Gracinda

This talk is based on:

[1] M. Dokuchaev and R. Exel, “Partial actions and subshifts”, arXiv:1511.00939.

Misha Dokuchaev

Given a finite alphabet Σ, consider the set of all infinite words in Σ, namely

$$\Sigma^{\mathbb{N}} = \Sigma \times \Sigma \times \Sigma \times \dots$$

This is a compact topological space under the product topology (Σ is given the discrete topology).

It is well known that $\Sigma^{\mathbb{N}}$ is homeomorphic to the Cantor set.

$$\Sigma^{\mathbb{N}} \simeq$$

We then define the shift map

$$S : \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}},$$

by

$$S(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots).$$

Definition.

(1) A subset $X \subseteq \Sigma^{\mathbb{N}}$ is said to invariant if $S(X) \subseteq X$.

(2) By a subshift we mean a pair of the form

$$(X, S|_X)$$

where $X \subseteq \Sigma^{\mathbb{N}}$ is invariant and closed (in the product topology).

Subshifts are often studied as dynamical systems, i.e. one would like to understand periodic points, invariant subsets, invariant measures, ergodicity etc.

Example 1. Choose a matrix $M = \{m_{a,b}\}_{a,b\in\Sigma}$, with $m_{a,b} \in \{0,1\}$, that is, a boolean matrix, and let X_M be the set consisting of all infinite words

$$a_1a_2a_3a_4\ldots$$

such that $m_{a_i,a_{i+1}} = 1$. Then X_M is invariant and closed, so we have a subshift. This is called a Markov subshift.

Example 2. Choose any set $\mathcal{F}$ of finite words, which we will call the forbidden words, and let $X_{\mathcal{F}}$ be the set of all infinite words which do not contain any forbidden subword. Then $X_{\mathcal{F}}$ is invariant and closed, so we again have a subshift.

For example, consider the usual alphabet

$$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l}, \mathtt{m}, \mathtt{n}, \mathtt{o}, \mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{s}, \mathtt{t}, \mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{x}, \mathtt{y}, \mathtt{z}\},$$

So we have infinite words of the form:

`portugalgottothenextstageofthesoccercompetitionxyxyxyxyhdgaebucbeeg...`

However, the word

`asdsdkdakhbraziloutofcopaamericavwrpprnrucnebscjshegdhejhvwxjhvwvv...`

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So we have infinite words of the form:

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However, the word

`asdsdkdakhbraziloutofcopaamericavwrpprnrucnebscjshegdhejhvwxjhvwvv...`

Should be forbidden!

Theorem. For every subshift X, there is a set $\mathcal{F}$ of finite words such that $X = X_{\mathcal{F}}$.

Notice that each word in $\mathcal{F}$ is finite, but $\mathcal{F}$ may contain infinitely many words.

Definition. A subshift X is said to be of finite type if $X = X_{\mathcal{F}}$ for some finite set $\mathcal{F}$. Otherwise we say that X is of infinite type.

Shifts of finite type are much easier to study. In particular if X is a shift of finite type then it is conjugated to a Markov subshift, meaning that there exists a homeomorphism

$$\varphi : X_M \to X,$$

such that

$$S|_X = \varphi(S|_{X_M})\varphi^{-1}.$$

(Remark: One may have to change the alphabet to achieve this).

One consequence of this is that all of the dynamical properties of X_M carry over to X.

Example 3. Billiards

A Billiard table can be any polygon (convex or not). The sides are labeled by the alphabet.

We consider all possible trajectories, but we only record the labels of the sides hit by the ball, such as

$$fdefab...$$

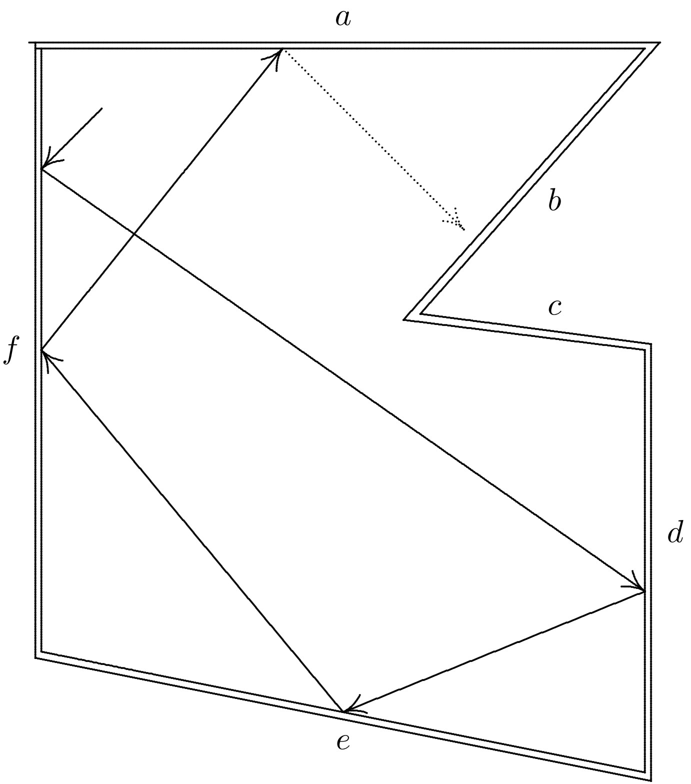
Some forbidden words in this example are:

$$aa$$

$$efe$$

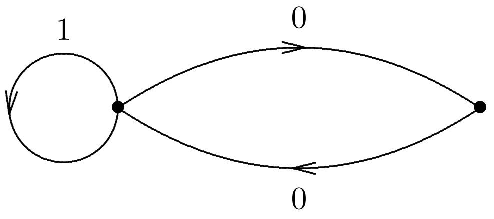
$$bc$$

The set X of all possible trajectories is a subshift, most likely of infinite type.



Example 4. Labeled graphs

Consider a directed graph, whose edges are labeled by the alphabet Σ.



We record the labels of all infinite paths, such as

$$1\underbrace{0000}_{4}111\underbrace{00000000}_{8}11\underbrace{000000}_{6}1\ldots$$

$$\underbrace{000}_{3}1\ldots$$

X is thus the set of all words arising as labels of infinite paths.

We do not care for the paths! Only the words they define matter.

The forbidden words are precisely:

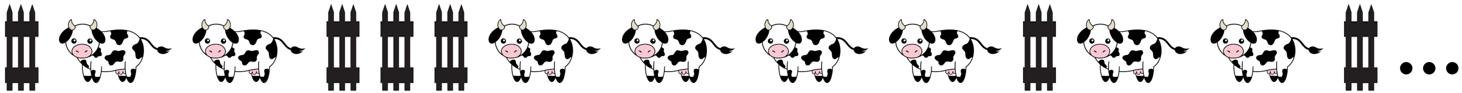
$$\mathcal{F} = \{1\underbrace{0000\ldots0}_{2n+1}1 : n \in \mathbb{N}\}$$

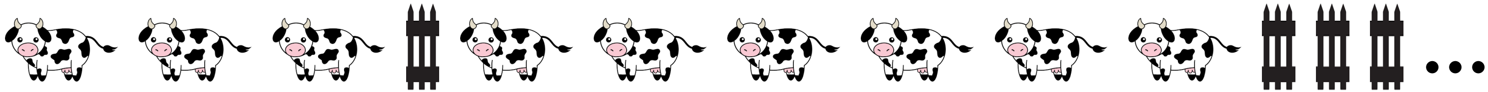
Definition.

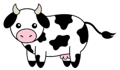
(1) For this specific example of labeled graph, X is called the even shift.

(2) A shift given by a labeled graph is called a sofic shift.

Sofic shifts are among the most tame of the shifts of infinite type.





0 =  1 = 

Always an even number of cows in each pen.

INTERMISSION

(Relations with partial semigroups)

Definition. Given any subshift X, the language of X is the set $Lang_X$ consisting of all finite words

$$\alpha = a_1 a_2 \dots a_n$$

occurring in some element of X, that is

$$x_1 \dots x_k \underbrace{a_1 a_2 \dots a_n}_{\alpha} y_1 y_2 \dots \in X$$

Given two words $\alpha, \beta \in Lang_X$, define a product

$$\alpha \cdot \beta = \begin{cases} \alpha\beta \quad \text{(concatenation)}, & \text{if } \alpha\beta \in Lang_X, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then this is a partial semigroup.

Regarding associativity, let X be the even shift and consider $\alpha = \text{‘11’}, \quad \beta = \text{‘000’}, \quad$ and $\quad \gamma = \text{‘11’}$. Then

$$\alpha\beta = 11000, \quad \text{and} \quad \beta\gamma = 00011$$

are both defined. However

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma = 1100011$$

is not defined!

This an an example of a partial semigroup failing to satisfy certain more demanding definitions of associativity.

From now on we Will fix an arbitrary subshift X. We want to algebrize X in the following sense: fix a field K, and let V be a K-vector space with a basis

$$\{\delta_x\}_{x\in X}$$

For each letter a in the alphabet Σ, consider the linear operators $\pi_a, \pi_a^* : V \to V$ given by

$$\pi_a(\delta_x) = \begin{cases} \delta_{ax}, & \text{if } ax \in X, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{Creation operator}$$

and

$$\pi_a^*(\delta_x) = \begin{cases} \delta_y, & \text{if } x = ay, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{Annihilation operator}$$

Our main object of study will be the algebra[1]

$$\mathcal{O}_X \;=\; \mathrm{Alg}\Big\langle \{\pi_a : a \in \Sigma\} \cup \{\pi_a^* : a \in \Sigma\} \Big\rangle \;\subseteq\; L(V)$$

This was first introduced and studied by Matsumoto starting in 1997 but there were serious mistakes in the early papers. In 2004 Carlsen and Matsumoto were able to fix some of these errors but only after changing the definition of $\mathcal{O}_X$. Overall there are now three different C*-algebras associated to X in the literature. The above is a purely algebraic adaptation of the currently accepted definition[2].

En passant, the multiplicative subsemigroup generated by the above set is an inverse semigroup. One might wonder which completion to a Boolean inverse semigroup one gets within $\mathcal{O}_X$. As it often happens, the answer is the tight completion.

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[1] Disclaimer: there are a few lies here, which we will ignore to avoid getting into further technicalities.

[2] Once you account for the lies.

SECOND INTERMISSION

(More on partial semigroups)

Define

$$\tilde{\pi} : \mathit{Lang}_X \to L(V)$$

as follows: given any $\alpha = a_1 \ldots a_n \in \mathit{Lang}_X$, put

$$\tilde{\pi}(\alpha) = \pi_{a_1} \circ \ldots \circ \pi_{a_n} \in L(V).$$

Then

$$\tilde{\pi}(\alpha)\tilde{\pi}(\beta) = \begin{cases} \tilde{\pi}(\alpha\beta), & \text{if } \alpha\beta \in \mathit{Lang}_X, \\ 0, & \text{otherwise.} \end{cases}$$

so $\tilde{\pi}$ is a representation of the partial semigroup Lang_X.

Problem. Find a general theory associating:

Any partial semigroup $S \longrightarrow$ some algebra $A(S)$ generated by S under natural relations,

such that $A(\mathit{Lang}_X) = \mathcal{O}_X$.

Warnings:

(1) The naive construction "KS" does not work!

(2) For infinite type subshifts you cannot use my semigroupoid techniques because of the failure of stronger associativity properties of Lang_X.

How to study $\mathcal{O}_X$? After all it is just a finitely generated algebra of linear transformations on a vector space.

For example, how to decide when it is simple (no nontrivial two sided ideals)?

Method: use partial actions and partial group representations.

For each letter a in the alphabet Σ, consider the subsets F_a and Z_a of X, given by

$$F_a = \{y \in X : ay \in X\}, \text{ and}$$

$$Z_a = \{x \in X : x \text{ begins with a}\}.$$

We then have the partially defined map

$$\theta_a : y \in F_a \to ay \in Z_a.$$

Want to define θ_g for every g in the free group $\mathbb{F} = \mathbb{F}(\Sigma)$.

When $g = a^{-1}$, for some $a \in \Sigma$, simply put

$$\theta_g = \theta_a^{-1}.$$

Given any other element g in $\mathbb{F}$, write

$$g = x_1 x_2 \ldots x_n$$

in reduced form, meaning that $x_i \in \Sigma \cup \Sigma^{-1}$, and $x_{i+1} \neq x_i$, and set

$$\theta_g = \theta_{x_1} \circ \theta_{x_2} \circ \ldots \circ \theta_{x_n}$$

For example, if $g = a^{-1}b$, where a and b are distinct elements of Σ, then

$$\theta_g = \theta_a^{-1} \circ \theta_b$$

This map wants to insert the letter b in front of a given x, and then remove the letter a. This is impossible! So θ_g is the empty map!

In fact θ_g is often the empty map, unless

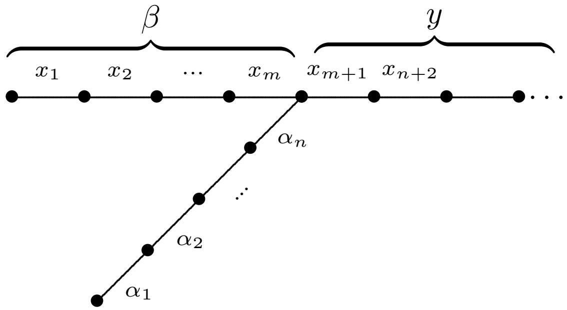
$$g = \alpha\beta^{-1},$$

with α and β being in the monoid generated by Σ, that is

$$g = \underbrace{a_1 a_2 \dots a_n}_{\alpha} \underbrace{b_m^{-1} \dots b_1^{-1}}_{\beta^{-1}},$$

with $a_n \neq b_m$. In this case $\theta_g(x)$ is defined if and only x begins with β, namely $x = \beta y$, and moreover $\alpha y \in X$. In this case one has

$$\theta_g(x) = \theta_{\alpha\beta^{-1}}(\beta y) = \alpha y$$



Proposition. The assignment $g \to \theta_g$ is a partial action of $\mathbb{F}$ on X.

However, there is a catch: normally partial actions on topological spaces are required to have open domain and ranges. However, F_a is not always open. In fact one has:

Proposition. The following are equivalent:

(1) F_a is open in X for every $a \in \Sigma$,

(2) X is a subshift of finite type,

(3) $S : X \to X$ is a local homeomorpism.

This is an indication that subshifts of infinite type involve hidden mysteries...

Moving from X to V=*span*(X), one may define for each g in $\mathbb{F}$, a linear operator $\pi_g : V \to V$, by

$$\pi_g(\delta_x) = \begin{cases} \delta_{\theta_g(x)}, & \text{if } \theta_g(x) \text{ is defined,} \\ 0, & \text{otherwise.} \end{cases}$$

Proposition. The assignment $g \to \pi_g$ is a partial representation of $\mathbb{F}$ on V, meaning that $\pi_1 = id_V$, and for every g and h in $\mathbb{F}$ one has:

$$\pi_g \pi_h \qquad = \pi_{gh}$$

and

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$$\pi_g \pi_h \pi_h^{-1} = \pi_{gh} \pi_h^{-1}$$

and

$$\pi_g^{-1} \pi_g \pi_h = \pi_g^{-1} \pi_{gh}$$

It is easy to see that $\mathcal{O}_X$ is also the algebra generated by $\{\pi_g : g \in \mathbb{F}\}$, so we see that:

$\mathcal{O}_X$ is an algebra generated by the range of a partial representation.

Many questions about such algebras are decided by looking at the commutative subalgebra

$$\mathcal{D}_X = \text{Alg}\Big\langle e_g = \pi_g \pi_g^{-1} = \text{projection onto the range of } \pi_g \Big\rangle \subseteq \mathcal{O}_X \subseteq L(V)$$

Recalling that the only nontrivial θ_g occur when $g = \alpha\beta^{-1}$, and that

$$\theta_g(\beta y) = \alpha y,$$

we have that e_g is the projection onto the subspace of V spanned by

$$\{\delta_{\alpha y} : y \in F_\alpha \cap F_\beta\}.$$

In particular, $\mathcal{D}_X$ is an algebra of diagonal operators, so in principle it should be possible to study it.

Once we understand the structure of $\mathcal{D}_X$, we will also understand a lot about $\mathcal{O}_X$, because

Theorem. $\mathcal{O}_X = \mathcal{D}_X \rtimes \mathbb{F}$ (partial crossed product).

It is well known that any unital commutative algebra generated by idempotents (such as $\mathcal{D}_X$ which is generated by the idempotents e_g) is isomorphic to

$$\mathcal{L}(\Omega) = \{\text{locally constant functions } f : \Omega \to K\},$$

where Ω is a compact, totally disconnected topological space.

The question is thus to find the topological space Ω_X such that

$$\mathcal{D}_X = \mathcal{L}(\Omega_X).$$

As in many duality theories, Ω_X is simply the spectrum of $\mathcal{D}_X$, namely the set of characters on $\mathcal{D}_X$, where by a character we mean an algebra homomorphism

$$\varphi : \mathcal{D}_X \to K.$$

A character is just pretending to be an evaluation map $\quad f \mapsto f(\omega), \;$ where $\omega \in \Omega_X$.

Notice that for every x in X, we have a character φ_x given by

$$\varphi_x(T) = \text{diagonal entry of } T \text{ associated to the basis vector } \delta_x.$$

So the correspondence $\quad \Phi : x \mapsto \varphi_x \quad$ gives a map from X into Ω_X.

Proposition. Φ is injective and its range is dense in Ω_X, so we may view $X \subseteq \Omega_X$.

Question. What exactly is Ω_X? Is it equal to X?

Proposition. X is a shift of finite type $\Leftrightarrow$ Φ is surjective $\Leftrightarrow$ Φ is a homeomorphism $\Leftrightarrow$ Φ is continuous.

Thus, if X is a subshift of finite type, we have $\mathcal{D}_X = \mathcal{L}(\Omega_X) = \mathcal{L}(X)$, so

$$\mathcal{O}_X = \mathcal{D}_X \rtimes \mathbb{F} = \mathcal{L}(X) \rtimes \mathbb{F}$$

For example, if X is the full shift on the alphabet $\Sigma = \{0, 1\}$, namely $X = \{0, 1\}^{\mathbb{N}}$, then $\mathcal{O}_X = \mathcal{O}_2$. Many properties about $\mathcal{O}_2$ can be derived from the description

$$\mathcal{O}_2 = \mathcal{L}\big(\{0, 1\}^{\mathbb{N}}\big) \rtimes \mathbb{F}_2,$$

plus the study of dynamical properties of the partial action of $\mathbb{F}_2$ on $\{0, 1\}^{\mathbb{N}}$. In particular, it is easy to see that this partial action is:

(1) Minimal (the orbit of every point is dense).

(2) Topologically free (points with trivial isotropy are dense).

A general result about partial actions says that (1) and (2) imply:

Theorem. $\mathcal{O}_2$ is simple.

A similar study can be done for every subshift of finite type, so the big question is what to do with subshifts which are not of finite type. The main trouble is to study the space Ω_X, which has caused a lot of headache!

Next Goal: to describe the space

$$\Omega_X = \mathrm{Spec}(\mathcal{D}_X) = \{\varphi : \mathcal{D}_X \to K,\ \varphi \text{ is an algebra homomorphism}\}$$

Since $\mathcal{D}_X$ is generated by the idempotents e_g, a homomorphism φ becomes completely characterized once we know the values of $\varphi(e_g)$, for every g in $\mathbb{F}$. Also, since e_g is idempotent, we must have that

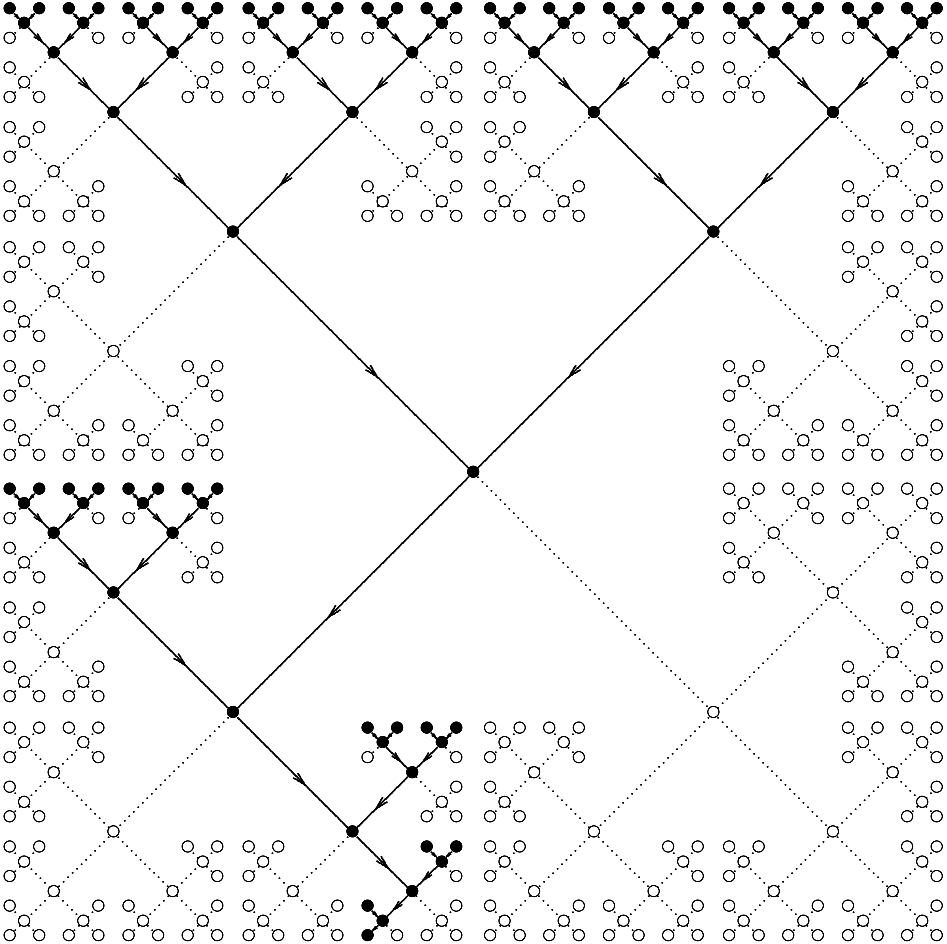
$$\varphi(e_g) \in \{0, 1\}, \quad \forall g \in \mathbb{F}.$$

So a character φ is encoded by the set

$$\xi_\varphi \;=\; \{g \in \mathbb{F} : \varphi(e_g) = 1\} \;\in\; \mathcal{P}(\mathbb{F}) \;=\; \text{set of all subsets of } \mathbb{F}$$

Consequently we have a model of Ω_X as a subset of $\mathcal{P}(\mathbb{F})$. That is, we may see

$$\Omega_X \subseteq \mathcal{P}(\mathbb{F}).$$



Given that

$$\Omega_X \subseteq \mathcal{P}(\mathbb{F}),$$

our task is to decide which ξ in $\mathcal{P}(\mathbb{F})$ actually come from a character.

The following is thus useful:

Proposition. For any ξ in Ω_X one has that:

(a) $1 \in \xi$,

(b) ξ is convex,

(c) for every $g \in \xi$, there exists a unique a in Σ, such that $ga \in \xi$,

(d) if $g \in \xi$, and α is a finite word such that $g\alpha \in \xi$, then α lies in the language of X.

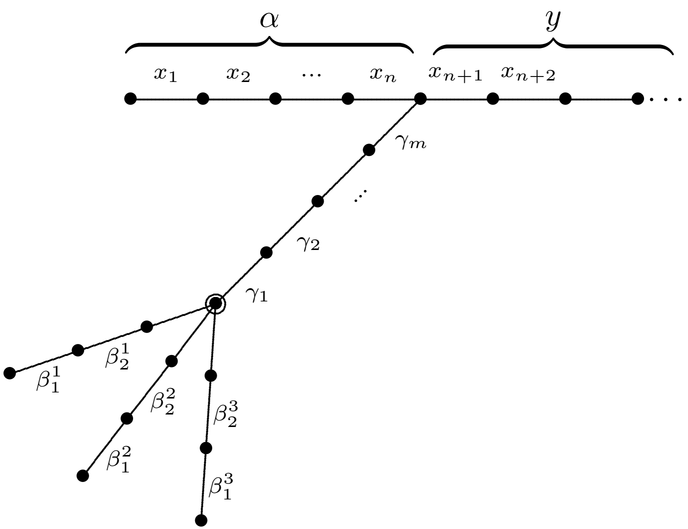
(e) $\xi \subseteq \mathbb{F}\mathbb{F}^{-1}$.

This is not a complete list, but knowing these and that $\Phi(X)$ is dense in Ω_X is often enough to answer most questions one would like to ask.

Theorem. Let X be a subshift. Then $\mathcal{O}_X$ is simple if and only if

(1) X contains at least one non eventually periodic infinite word (i.e. a point not of the form $\alpha\gamma\gamma\gamma\ldots$) and

(2) for every x in X, and for every finite set $\{\beta_1, \ldots .\beta_n\}$ with $\cap F_{\beta_i} \neq \emptyset$, there are finite words α and γ such that $x = \alpha y$, and $\beta_i\gamma y \in X$, for every i. Moreover, $|\alpha| + |\gamma|$ should be bounded by some number depending on the β_i, but not on x.



Proof. Use the above description of $\mathcal{O}_X$, and prove that the partial action of $\mathbb{F}$ on it is topologically free and minimal.

Thank you!