# The soluble kernel of a finite semigroup is computable

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Open problem

Is the soluble kernel of a finite semigroup computable?

Coulbois, Sapir and Weil claimed in 2003: "the solution of this difficult open question would have interesting consequences in finite monoid theory and in computational complexity"

[CSW03] T. Coulbois, M. Sapir and P. Weil, *A note on the continuous extensions of injective morphisms between free groups to relatively free profinite groups*, Publ. Mat., 47 (2003), 477–487.

[MSW01] S. Margolis, M. Sapir and P. Weil, *Closed subgroups in pro-**V** topologies and the extension problem for inverse automata*, Int. J. Algebra and Comput., Vol. 11, No. 4 (2001), 405–445.

[ST01] H. Straubing and B. Thérien, *Regular languages defined by generalized first-order formulas with a bounded number of bound variables*, in: "*STACS 2001*" (Dresden), Lecture Notes in Comput. Sci. 2010, Springer, Berlin, 2001, pp. 551–562.

Recently, B. Steinberg published a post in the following mathoverflow website:
http://mathoverflow.net/questions/156761/computing-the-pro-solvable-closure-of-a-finitely-generated-subgroup-of-a-free-gr

**Theorem A**

The soluble kernel of a finite semigroup is computable

**Definition**

Let $S$ be a semigroup. Then, *the kernel of $S$* is the intersection of the kernels of all relational morphisms from $S$ to a group $G$. This subsemigroup is denoted by $\mathrm{K}(S)$.

This notion was first introduced by J. Rhodes and B. Tilson, using the terminology of "Type II elements", in their 1972 seminar paper:

[RT72] J. Rhodes and B. R. Tilson, *Improved lower bounds for the complexity of finite semigroups*, J. Pure Appl. Algebra, 2 (1972) 13–71.

**Theorem ([RT72])**

*The regular elements of the kernel are computable.*

Theorem (Type II Theorem)

*The kernel of every semigroup is computable.*


[Ash91] C. J. Ash, *Inevitable graphs: a proof of the type II conjecture and some related decision procedures*, Int. J. Algebra Comput., 1 (1991), 127–146.

[RZ93] L. Ribes and P. Zalesskiĭ, *On the profinite topology on a free group*, Bull. London Math. Soc., 25 (1993), 37–43.

[PR91] J.-É. Pin and C. Reutenauer, *A conjecture on the Hall topology for the free group*, Bull. London Math. Soc., 23 (1991), 356–362.

## Consequences of Type II Theorem

[M91] S. W. Margolis, *Consequences of Ash's proof of the Rhodes Type II Conjecture*, Proceedings of the Monash Conference on Semigroup Theory, World Scientific, Singapore (1991), 180–205.

[HMPR91] K. Henckell, S. W. Margolis, J.-É. Pin and J. Rhodes, *Ash's type II theorem, profinite topology and Malcev products: Part I*, Int. J. Algebra and Comput., 1 (1991), 411–436.

## Definition

Let **V** be a variety of groups and $S$ be a semigroup. Then:

$$\mathrm{K}_{\mathbf{V}}(S) = \bigcap \{\tau^{-1}(1) : \tau\colon\ S \multimap G \in \mathbf{V}\ \}$$

is a subsemigroup of $S$, called the **V**-*kernel* of $S$.

**Open problem**

For which varieties of groups $\mathbf{V}$ is the $\mathbf{V}$-kernel of a semigroup computable?

- $\mathbf{V}_p = \{G : G \text{ is a } p\text{-group}\}$, for every $p \in \mathbb{P}$.

  [RZ94] L. Ribes and P. Zaleskiĭ, *The pro-p topology of a free group and algorithmic problems in semigroups*, Int. J. Algebra Comput., 4 (1994), 359–374.

- $\mathbf{Ab} = \{G : G \text{ abelian group}\}$.

  [D98] M. Delgado, *Abelian pointlikes of a monoid*, Semigroup Forum 56 (1998), 339–361.

- **V** every variety of abelian groups with decidable membership problem and which generates the abelian group variety.

  [S99] B. Steinberg, *Monoid kernels and profinite topologies on the free abelian group*, Bull. Austral. Math. Soc., 60 (1999), 391–402.

- $\mathbf{N} = \{G : G \text{ nilpotent group}\}$.

  [ASS15] J. Almeida, M. H. Shahzamanian and B. Steinberg *The pro-nilpotent group topology on a free group*, arXiv:1511.01947, 2015.

## Ingredients of the proof of Theorem A

- Theorem in [RZ94]: If **V** is an extension-closed variety of groups, the finite product of $n$ pro-**V** closed finitely generated subgroups in a free group is pro-**V** closed. As a consequence, in the case of soluble groups **S**, if the pro-**S** closure of a finitely generated subgroup is computable, then the **S**-kernel is computable.
- Theorem in [MSW01]: If **V** is an extension-closed variety of groups, computing the pro-**V** closure of a finitely generated subgroup of a free group is equivalent to computing the **V**-extensible closure of a finitely generated subgroup of a free group. As a consequence, if the regular elements of the **S**-kernel are computable, then the pro-**S** closure of a finitely generated subgroup is computable. Therefore, the computability of the regular elements of the soluble kernel implies the computability of the soluble kernel.

A. Ballester-Bolinches, R. Esteban-Romero and V. Pérez-Calabuig, *On generalised kernels of finite semigroups*, submitted.

## Reduction theorem

- A reduction theorem about the computability of the elements of $\mathrm{K}_{\mathbf{V}}(S) \cap J$, with $J$ regular $\mathcal{J}$-class.
- It is possible to consider $J$, 0-minimal in $S$, so that $J^0 = \mathcal{M}^0(G, A, B, C)$. Then we can construct a group $G_0 \in \mathbf{V}$, which is a quotient of G, and a set of indices $\Lambda_0$, such that $S$ acts partially injectively on the set $X = \{(i, g) : i \in \Lambda_0, g \in G_0\}$.

## Reduction theorem

- We can construct a semigroup of partial one-to-one transformatrions, $U$, with a unique 0-minimal $\mathcal{J}$-class $\bar{J}$, such that $\bar{J}^0 = \mathcal{M}^0(G_0, \Lambda_0, \Lambda_0, I_{r_0})$ is a Brandt semigroup ($r_0 = |\Lambda_0|$).
- $\mathsf{K}_{\mathbf{V}}(S) \cap J$ is computable if, and only if, $\mathsf{K}_{\mathbf{V}}(U) \cap \bar{J}$ is computable.

- We can suppose $U$ $\tilde{A}$-generated, where $\tilde{A} = A \cup A^{-1}$ and $A$ is an alphabet.
- We can construct an inverse $A$-graph, $\Gamma_U$, such that $U$ is its inverse transition monoid. Note that $V(\Gamma_U) = X$.

## Theorem

*Let $\Gamma$ be an inverse $A$-graph. Let us consider in $V(\Gamma)$ the relation, $v \sim v'$ if there exists $x \in \mathsf{K}_{\mathbf{V}}(M(\mathcal{I}(\Gamma))$ such that $vx = v'$. Then, $\sim$ is the least inverse $A$-graph congruence in $\Gamma$, such that $\Gamma/\sim$ is $\mathbf{V}$-extensible.*

[S01] B. Steinberg, *Finite state automata: a geometric approach*, Trans. Amer. Math. Soc, Vol. 353, No. 9 (2001), 3409–3464

## Theorem

*Let us consider in $V(\Gamma_U) = X$ the relation, $(i, g) \sim (i', g')$ if, and only if, $(i, g^{-1}g', i') \in \mathsf{K}_{\mathbf{V}}(U) \cap \bar{J}$. Then, $\sim$ is the least inverse $A$-graph congruence in $\Gamma_U$, such that $\Gamma_U/\sim$ is $\mathbf{V}$-extensible.*

## Proposition

We can denote such least inverse $A$-graph congruence by $\sim_{\mathbf{V}}$. Then $\sim_{\mathbf{V}}$ is computable if, and only if, $\mathsf{K}_{\mathbf{V}}(U) \cap \bar{J}$ is computable.

## Theorem

*Let $\sim_{\mathbf{V}}$ be the least inverse A-graph congruence in $\Gamma_U$ such that $\Gamma_U$ is $\mathbf{V}$-extensible in $\Gamma_H$, with $M(\Gamma_H) = H \in \mathbf{V}$. Then, there exists a group $G$ and $\Gamma_G$ with $M(\Gamma_G) = G$, such that the following assertations hold:*

1. *$\Gamma_U$ is embedded in $\Gamma_G$.*
2. *If we consider in $\Gamma_G$ the relation in $V(\Gamma_G)$ given by: $v \sim'_{\mathbf{V}} v'$ if, and only if, there exists an element $g \in G^{\mathbf{V}}$ such that $vg = v'$. Then, $\Gamma_G / \sim'_{\mathbf{V}} = \Gamma_H$.*
3. *In particular, $\sim_{\mathbf{V}} = (\sim'_{\mathbf{V}})|_{V(\Gamma_U) \times V(\Gamma_U)}$.*

**Definition**

Given an equivalence relation $\sim$ in $V(\Gamma_U)$, we denote by $\overline{\sim}$ the least inverse $A$-graph congruence in $\Gamma_U$ which contains $\sim$.

Let $S$ be a semigroup. For each $n \in \mathbb{N}$, we define:

$$\mathsf{K}^n_{\mathbf{V}}(S) = \mathsf{K}_{\mathbf{V}}(\mathsf{K}^{n-1}_{\mathbf{V}}(S))$$

There exists $n \in \mathbb{N}$ for wich $\mathsf{K}^n_{\mathbf{V}}(S) = \mathsf{K}^{n+1}_{\mathbf{V}}(S)$. We denote by $K^{\omega}_{\mathbf{V}}(S)$ such subsemigroup.
Then, for every $n \in \mathbb{N}$, we define in $V(\Gamma_U) = X$ the equivalence relation:

$$(i, g) \sim^n_{\mathbf{V}} (i', g') \text{ if, and only if, } (i, g^{-1}g', i') \in \mathsf{K}^n_{\mathbf{V}}(U) \cap \bar{J}$$

### Theorem

*Let* $\mathbf{V}$ *be an extension-closed variety of groups. Then, in* $\Gamma_U$*, it holds:*

$$\overline{\sim^n_{\mathbf{V}}} = \sim_{\mathbf{V}} \quad \text{for every } n \in \mathbb{N}$$

### Theorem

*Let* $\mathbf{S}$ *be the variety of soluble groups. Let* $\mathbf{N}$ *and* $\mathbf{Ab}$ *be the variety of, respectively, nilpotent and abelian groups. Then, in* $\Gamma_U$*, it holds:*

$$\sim_{\mathbf{S}} = \overline{\sim^{\omega}_{\mathbf{N}}}, \quad \sim_{\mathbf{S}} = \overline{\sim^{\omega}_{\mathbf{Ab}}}$$

### Corollary

*Consequently,* $\sim_{\mathbf{S}}$ *is computable in* $\Gamma_U$ *and therefore,* $K_{\mathbf{S}}(U) \cap \bar{J}$ *is computable.*