Combinatorics of cyclic shifts in the plactic, hypoplactic, sylvester, and related monoids

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Young tableaux & the plactic monoid

Let $n \in \mathbb{N}$ and let $A_n = \{1 < 2 < 3 < \ldots < n\}$.

1	1	2	3	5	7
3	4	6			
5	5	7			
6					

- Rows non-decreasing left to right
- Columns increasing top to bottom
- Left-justified, longer rows on top

Schensted's algorithm computes a tableau $P(u)$ from a word $u \in A_n^*$. Define

$$u \equiv_{\text{plac}} v \iff P(u) = P(v).$$

Theorem (Knuth 1970)

The relation $\equiv_{\text{plac}}$ is a congruence on A_n^*.

The factor monoid $\text{plac}_n = A_n^*/{\equiv_{\text{plac}}}$ is the Plactic monoid of rank n

- Connected with combinatorics, quantum groups, symmetric functions, representations of $\mathfrak{sl}_n$ and $\mathfrak{S}_n$.

‘Plactic-like’ monoids

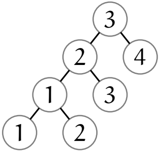
Plactic monoid
plac_n
Young tableau

1	1	2	3
2	3	4	
3			

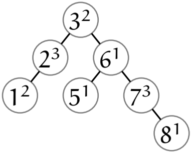
Hypoplactic monoid
hypo_n
Quasi-ribbon tableau

1	1				
	2	2	3	3	3
					4

Sylvester monoid
sylv_n
Binary search tree



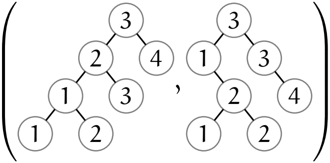
Taiga monoid
taig_n
BST with multiplicities



Stalactite monoid
stal_n
Stalactite tableau

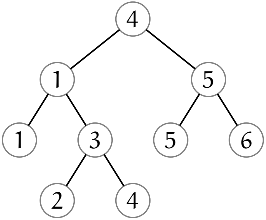
1	2	4	3
1	2		3
	2		3

Baxter monoid
baxt_n
Pair of twin binary search trees



Binary search trees & sylvester monoid

Binary search tree (BST):



To insert x into a BST T:

- Add x as a leaf node in the unique position that yields a BST.

For a word $u = u_k u_{k-1} \cdots u_1 \in A_n^*$.

- Start with an empty BST and insert u_1, then u_2, ..., finally u_n.
- Call the resulting BST $\mathcal{T}(u)$.

Define $\equiv_{\text{sylv}}$ on A_n^* by $u \equiv_{\text{sylv}} v \iff \mathcal{T}(u) = \mathcal{T}(v)$.

Theorem (Hivert et al. 2005)

The relation $\equiv_{\text{sylv}}$ is a congruence on A_n^*.

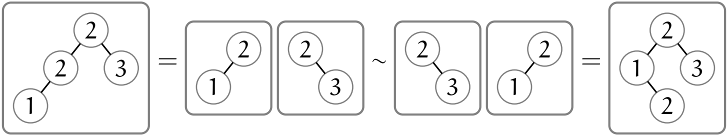
The factor monoid $\text{sylv}_n = A_n^*/\equiv_{\text{sylv}}$ is the sylvester monoid of rank n

Cyclic shifts

Define the cyclic shift relation as follows: for $u, v \in M$,

$$u \sim v \iff (\exists x, y \in M)(u = xy \wedge v = yx).$$

For example, in sylv_n:



$$\mathcal{T}(1232) = \mathcal{T}(12)\mathcal{T}(32) \sim \mathcal{T}(32)\mathcal{T}(12) = \mathcal{T}(3212)$$

- For groups, $\sim$ is the usual conjugacy relation.
- For free monoids, $\sim$ relates words that differ by a cyclic shift.
- $\sim$ is not in general transitive.

Cyclic shifts in ‘plactic-like’ monoids

Define the evaluation or content function ev : $A_n^* \to \mathbb{N}_0^n$ by

$$\mathrm{ev}(w) = \big(|w|_1, |w|_2, \ldots, |w|_n\big).$$

For example, $\mathrm{ev}(23111341) = (4, 1, 2, 1)$.

- If $u \equiv_{\mathrm{plac}} v$, or $u \equiv_{\mathrm{hypo}} v$, or $u \equiv_{\mathrm{sylv}} v$, ... then $\mathrm{ev}(u) = \mathrm{ev}(v)$. (That is, plac_n, hypo_n, sylv_n, ... are multihomogeneous.)
- So it makes sense to define ev : $\mathrm{plac}_n \to \mathbb{N}_0^n$ etc.

Theorem (Lascoux & Schützenberger 1981)

Let $s, t \in \mathrm{plac}_n$. If $\mathrm{ev}(s) = \mathrm{ev}(t)$, then $s \sim^* t$.

Corollary

Let $s, t \in \mathrm{hypo}_n$. If $\mathrm{ev}(s) = \mathrm{ev}(t)$, then $s \sim^* t$.

Theorem (C, Malheiro 2015)

Let $s, t \in \mathrm{sylv}_n$. If $\mathrm{ev}(s) = \mathrm{ev}(t)$, then $s \sim^* t$.

Corollary

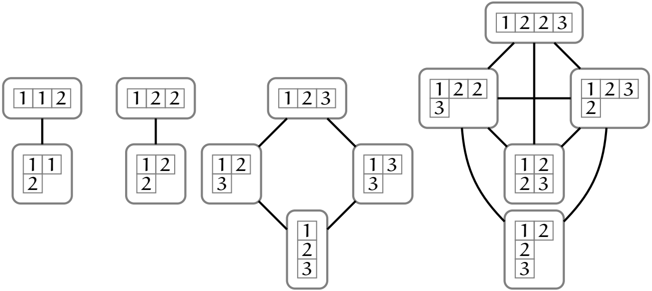
Let $s, t \in \mathrm{taig}_n$. If $\mathrm{ev}(s) = \mathrm{ev}(t)$, then $s \sim^* t$.

Cyclic shift graph

For a monoid M, define the cyclic shift graph $\Gamma(M)$ to be the graph with

- vertex set M
- an edge from u to v whenever $u \sim v$.

Part of $\Gamma(\text{plac}_3)$:



Corollary

A given connected component C of $\Gamma(\text{plac}_n)$ (or $\Gamma(\text{hypo}_n)$ or $\Gamma(\text{sylv}_n)$ or $\Gamma(\text{taig}_n)$) consists of precisely the vertices with some evaluation γ_C.

Diameters of connected components

Theorem (Choffrut & Mercaş 2013)

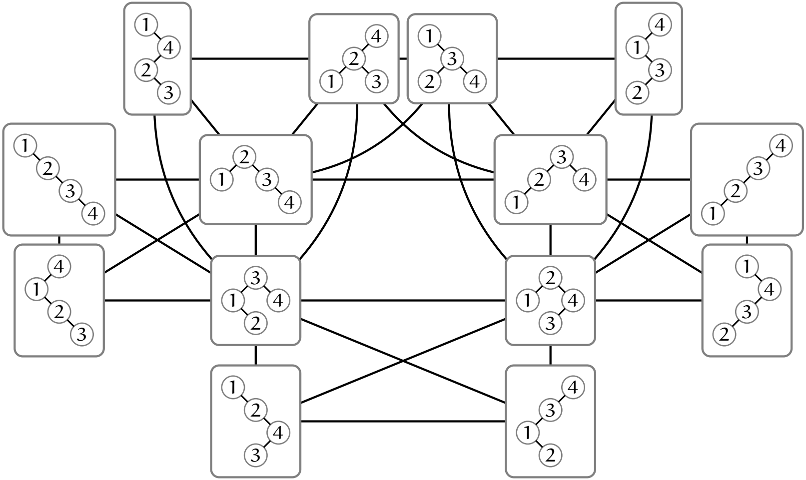
Let $s, t \in \text{plac}_n$. If $\text{ev}(s) = \text{ev}(t)$, then $s \sim^{\leqslant 2n-2} t$.

Corollary

The diameter of a connected component of $\Gamma(\text{plac}_n)$ is less than or equal to $2n - 2$.

- What is the maximum diameter of a connected component of $\Gamma(\text{plac}_n)$?
- What about $\Gamma(\text{hypo}_n)$, $\Gamma(\text{sylv}_n)$, $\Gamma(\text{taig}_n)$, …?

Conjecture from experiments using Sage

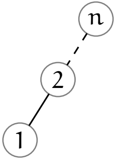
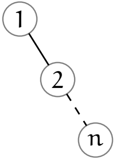


Conjecture

In $\Gamma(\mathrm{plac}_n)$, $\Gamma(\mathrm{hypo}_n)$, $\Gamma(\mathrm{sylv}_n)$, $\Gamma(\mathrm{taig}_n)$, maximum diameter of a connected component is $n-1$.

Lower bounds

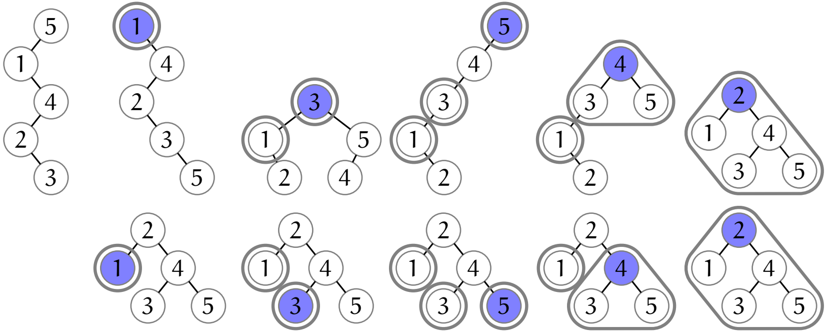
In sylv_n, at least $n-1$ applications of $\sim$ separate

$\mathcal{T}(12\cdots n) =$  and $\mathcal{T}(n\cdots 21) =$  .

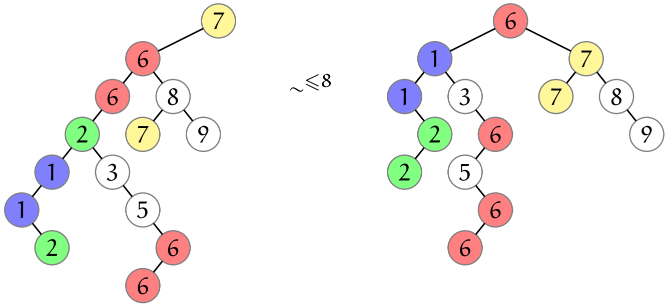
- So the connected component of these elements in $\Gamma(\text{sylv}_n)$ has diameter at least $n-1$.
- Similarly for plac_n, hypo_n, taig_n, ...

Upper bound example: sylv_5

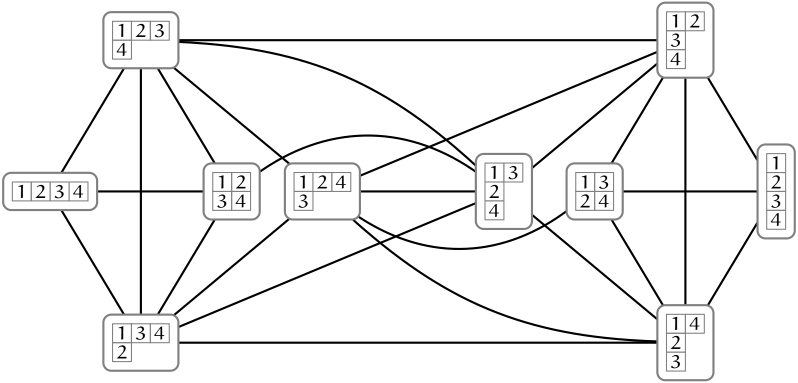
$$\begin{aligned}
\mathcal{T}(32415) &\sim \mathcal{T}(53241) \\
&= \mathcal{T}(53241) \sim \mathcal{T}(24153) \\
&= \mathcal{T}(45213) \sim \mathcal{T}(21345) \\
&= \mathcal{T}(21345) \sim \mathcal{T}(52134) \\
&= \mathcal{T}(21354) \sim \mathcal{T}(13542)
\end{aligned}$$



Binary search trees with repeated elements



Upper bounds



- For plac_n, best upper bound is $2n - 3$, using a slight improvement to Choffrut–Mercaş proof.

Summary

Monoid	Components char. by ev	Max. diameter in rank n	
Plactic	Y	$n-1 \leqslant ? \leqslant 2n-3$	(Conj. $n-1$)
Hypoplactic	Y	$n-1$	
Sylvester	Y	$n-1$	
Taiga	Y	$n-1$	
Stalactite	N	3	
Baxter	N	?	

- There exist finitely generated multihomogeneous monoids where there is no bound on the size of connected components.