The κ-word problem over pseudovarieties of the form DRH

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CSA - Celebrating the 60th birthday of Jorge Almeida and Gracinda Gomes

What is a pseudovariety?

A pseudovariety is a class of finite semigroups that is closed under taking subsemigroups, homomorphic images, and finite direct products.

Example

S: consists of all finite semigroups.

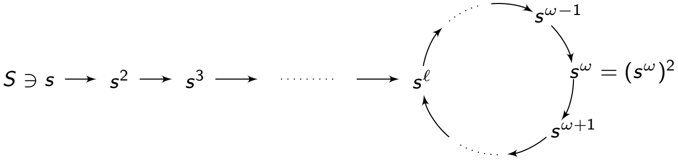
Why studying pseudovarieties?

Eilenberg's correspondence

varieties of rational languages	$\longleftrightarrow$	pseudovarieties of finite semigroups
combinatorial properties		algebraic properties

The κ-word problem over a pseudovariety V

We use κ to denote the implicit signature consisting of multiplication and $(\omega - 1)$-power.



Fix an alphabet A. A κ-word is an element of the free unary semigroup on A over the signature κ.

Given a pseudovariety V, solving the κ-word problem over V consists in deciding whether two given κ-words have the same natural interpretation on every semigroup of V.

What does DRH mean?

Given a semigroup S, two elements $s, t \in S$ are said to be $\mathcal{R}$-equivalent if s is a prefix of t and t is a prefix of s.

What does DRH mean?

Given a pseudovariety of groups H, DRH is the pseudovariety of all finite semigroups whose regular $\mathcal{R}$-classes lie in H.

Particular case: the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups.

Almeida, Zeitoun'2007: solved the κ-word problem over R.

Schützenberger'1976/77: identified the varieties of rational languages associated with pseudovarieties of the form DRH under Eilenberg's correspondence.

Almeida, Weil'1997: described the structure of free pro-DRH semigroups.

- $\overline{\Omega}_A \mathsf{V}$: free A-generated pro-V semigroup.
  Its elements are called pseudowords over V.

- $\Omega_A^\kappa \mathsf{V}$: κ-subalgebra of $\overline{\Omega}_A \mathsf{V}$ generated by A.
  Its elements are called κ-words over V.

- $\rho_{\mathsf{V}} : \overline{\Omega}_A \mathsf{W} \to \overline{\Omega}_A \mathsf{V}$: natural projection whenever $\mathsf{V} \subseteq \mathsf{W}$.

- $c(\_)$: content function, that is, the projection $\rho_{\mathsf{SI}}(\_)$ whenever $\mathsf{SI} \subseteq \mathsf{W}$.

Since every κ-word has a natural interpretation on every finite semigroup, it uniquely determines an element of $\Omega_A^\kappa \mathsf{S} \subseteq \overline{\Omega}_A \mathsf{S}$.

Thus, to solve the κ-word problem over DRH amounts to decide if two elements $u, v \in \Omega_A^\kappa \mathsf{S}$ are such that $\rho_{\mathsf{DRH}}(u) = \rho_{\mathsf{DRH}}(v)$.

On the structure of free pro-DRH semigroups

Uniqueness of the left basic factorization over DRH

Let $w \in \overline{\Omega}_A\mathsf{DRH}$. Then, there is a unique factorization of the form

$$w = w_\ell \cdot a \cdot w_r$$

satisfying $c(w) = c(w_\ell) \uplus \{a\}$.

Such factorization is called the left basic factorization of w.

On the structure of free pro-DRH semigroups

We may iterate the left basic factorization of w to the right as follows:

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$$w = w_1 a_1 \cdot w_2 a_2 \cdot w_3 a_3 \cdot w_3'$$

On the structure of free pro-DRH semigroups

We may iterate the left basic factorization of w to the right as follows:

$$w = w_1 a_1 \cdot w_2 a_2 \cdot w_3 a_3 \cdots w_k a_k \cdot w'_k$$

On the structure of free pro-DRH semigroups

We may iterate the left basic factorization of w to the right as follows:

$$w = w_1 a_1 \cdot w_2 a_2 \cdot w_3 a_3 \cdots w_k a_k \cdot w'_k$$

We write $\mathrm{lbf}_k(w) = w_k a_k$, when defined.

Definitions

The cumulative content of w, denoted $\vec{c}(w)$, is

- the empty set if the above iteration stops;
- the ultimate value of $c(w'_k)$, otherwise.

The regular part of w is $\mathrm{reg}(w) = w'_m$, where m is the least integer such that $\vec{c}(w) = c(w'_m)$.

On the structure of free pro-DRH semigroups

Proposition (Almeida, Weil'1997)

Let $w \in \overline{\Omega}_A\mathsf{DRH}$ be such that $\vec{c}(w) = c(w)$. Then,

- all the accumulation points of the sequence $(\mathrm{lbf}_1(w) \cdots \mathrm{lbf}_k(w))_{k \geq 1}$ belong to the same $\mathcal{R}$-class, which is regular;
- if R is a regular $\mathcal{R}$-class of $\overline{\Omega}_A\mathsf{DRH}$ (and hence, a group), and e is its identity, then $\rho_{\mathsf{H}}|_R : R \to \overline{\Omega}_{c(e)}\mathsf{H}$ is a homeomorphism.

Corollary

Let u, v be pseudowords over DRH. Then,

$$u = v \text{ if and only if } u \mathrel{\mathcal{R}} v \text{ and } \mathrm{reg}(u) = \mathrm{reg}(v) \text{ modulo } \mathsf{H}.$$

THEOREM

A pseudovariety of groups H has decidable κ-word problem if and only if the same happens for the pseudovariety DRH.

Idea of proof of the "if" part. Let $u, v \in \Omega_A^\kappa \mathsf{S}$. Then,

$$u = v \text{ modulo H} \iff (uv)^\omega u = (uv)^\omega v \text{ modulo DRH}.$$

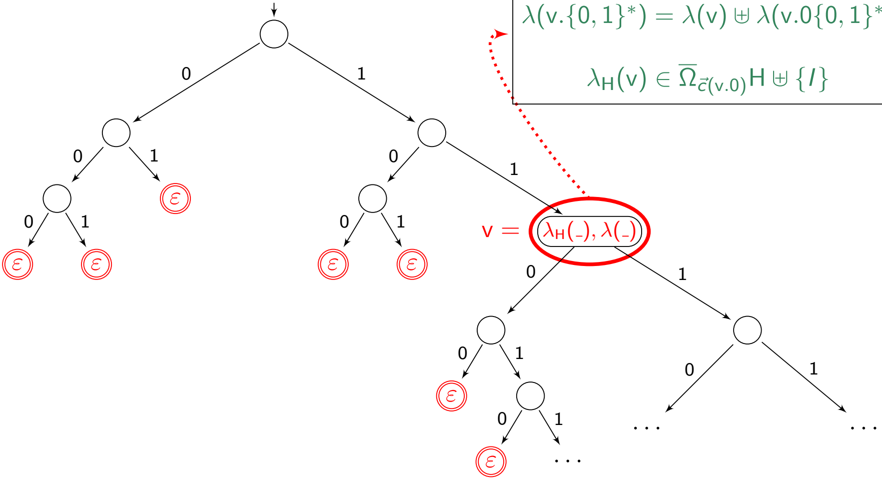
Idea of proof of the "only if" part. On the next slides...

A-labeled DRH-trees

$\lambda(\mathsf{v}) \in A$ is such that

$$\lambda(\mathsf{v}.\{0,1\}^*) = \lambda(\mathsf{v}) \uplus \lambda(\mathsf{v}.0\{0,1\}^*)$$

$$\lambda_{\mathsf{H}}(\mathsf{v}) \in \overline{\Omega}_{\vec{c}(\mathsf{v}.0)}\mathsf{H} \uplus \{I\}$$



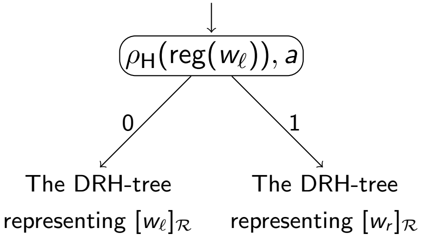
THEOREM

There exists a bijection

$$\pi : \{\mathcal{R}\text{-classes of } \overline{\Omega}_A\mathsf{DRH}\} \to \{A\text{-labeled } \mathsf{DRH}\text{-trees}\}.$$

Idea of proof.

Let $w \in \overline{\Omega}_A\mathsf{DRH}$, with left basic factorization given by $w = w_\ell \cdot a \cdot w_r$.



□

Theorem

There exists a bijection

$$\pi : \{\mathcal{R}\text{-classes of } \overline{\Omega}_A\mathsf{DRH}\} \to \{A\text{-labeled } \mathsf{DRH}\text{-trees}\}.$$

Fact: If w is a κ-word whose left basic factorization is given by $w = w_\ell \cdot a \cdot w_r$, then both w_ℓ and w_r are κ-words.

If w is a κ-word over DRH and v is a node of $\pi([w]_{\mathcal{R}})$, then the label $\lambda_{\mathsf{H}}(\mathsf{v})$ is a κ-word as well.

Example - constructing a DRH-tree

$abcab(a^{\omega+1}b)^{\omega}$

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$ab \cdot c \cdot ab(a^{\omega+1}b)^{\omega}$

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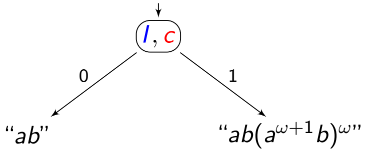

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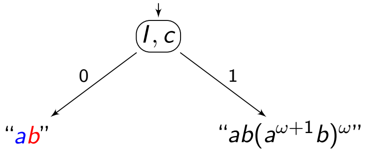

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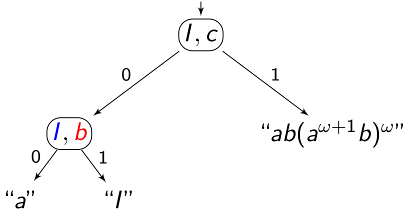

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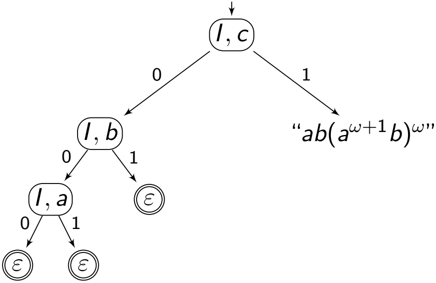

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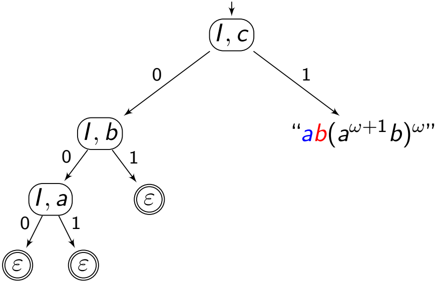

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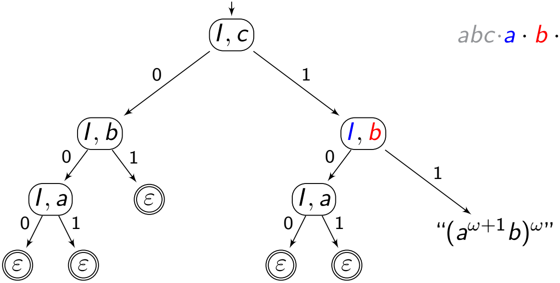

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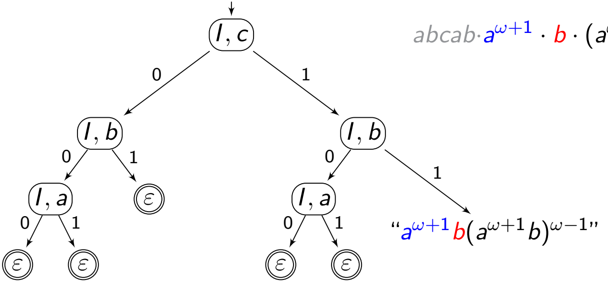
Example - constructing a DRH-tree

$abcab \cdot a^{\omega+1} \cdot b \cdot (a^{\omega+1}b)^{\omega-1}$



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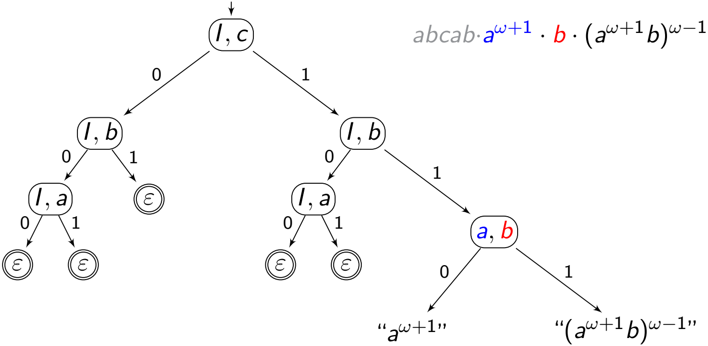

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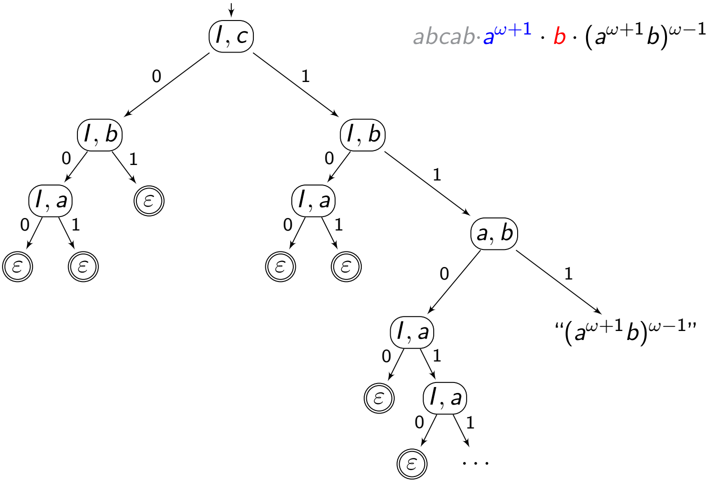

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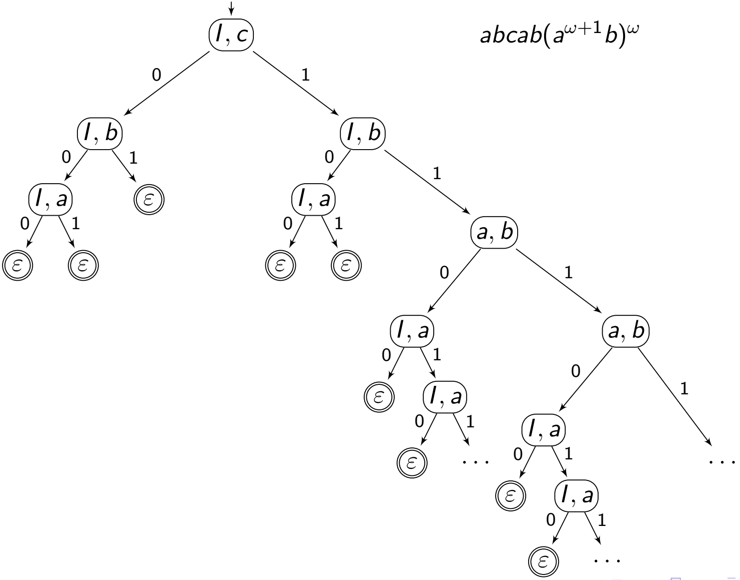

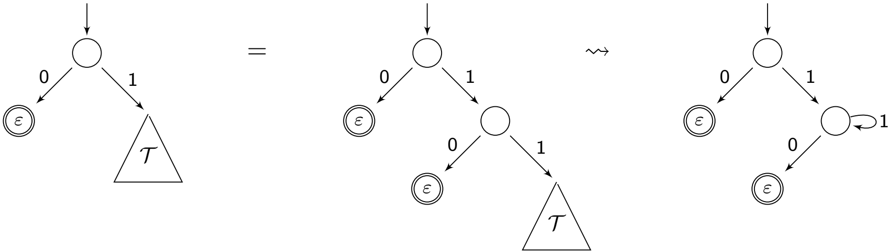

Proposition

Let $\mathcal{T}$ be a DRH-tree that represents the $\mathcal{R}$-class of a κ-word over DRH. Then, one may obtain a finite structure by identifying pairs of states of $\mathcal{T}$ that are "equivalent".

Example - wrapping a DRH-tree

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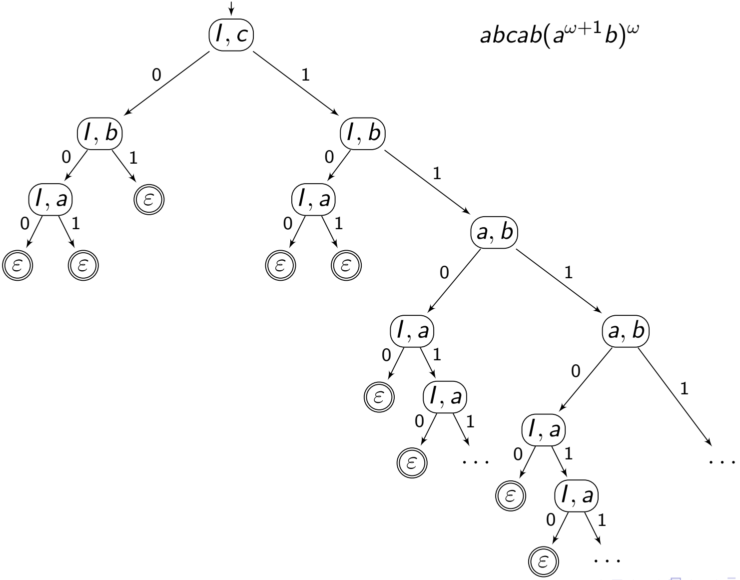

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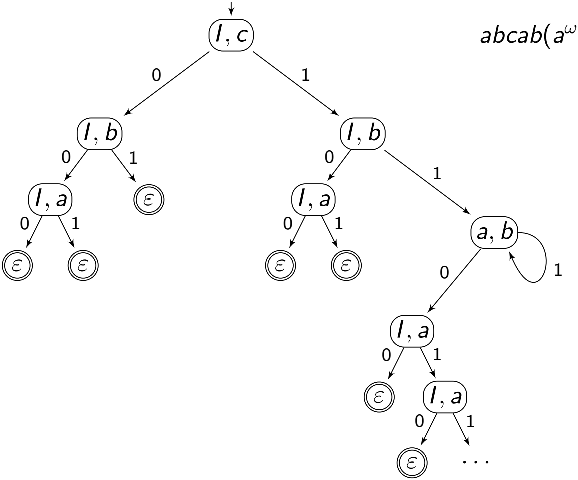

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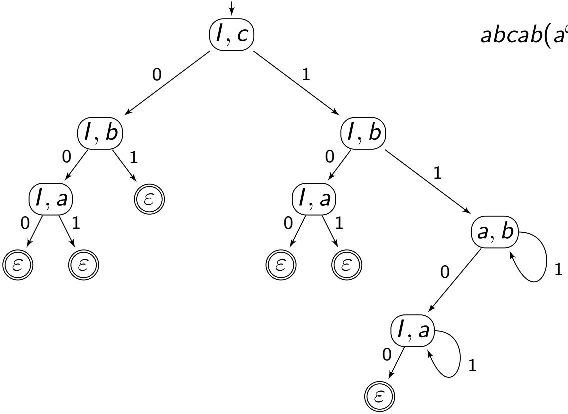
EXAMPLE - WRAPPING A DRH-TREE



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HOW TO SOLVE THE κ-WORD PROBLEM OVER DRH?

Let u and v be κ-words.

1. Compute finite DRH-automata representing the $\mathcal{R}$-classes of u and v.

2. Compare these automata. Do they represent the same $\mathcal{R}$-class?

(a) No: The κ-words u and v do not represent the same element over DRH;

(b) Yes: Solve the κ-word over H for u and w.

Examples of applications

- The pseudovariety DRAb has decidable κ-word problem.
- Let p be a prime number. If $\mathsf{H} \supseteq \mathsf{G}_p$ is a pseudovariety of groups, then DRH has decidable κ-word problem.
  (Baumslag'1965)

Thank you!