On the finite basis problem for deformed diagram monoids and related monoids

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Outline

- Identity bases of algebraic structures
- Deformed diagram monoids
- Related monoids
- Involutions

An identity (= law) of an algebraic structure $\mathfrak{M}$ is a formal equality $u = v$ of two terms u and v (in the language of $\mathfrak{M}$) which is identically true in $\mathfrak{M}$.
An identity base B for $\mathfrak{M}$ is a set of identities of $\mathfrak{M}$ so that **all** identities of $\mathfrak{M}$ can be derived from B.

Example: $(\mathbb{N}, +, \cdot, 1)$; an identity base is:

1. $x + y = y + x$
2. $x + (y + z) = (x + y) + z$
3. $1 \cdot x = x$
4. $x \cdot y = y \cdot x$
5. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
6. $x \cdot (y + z) = x \cdot y + x \cdot z$

Example extended: $(\mathbb{N}, +, \cdot, \uparrow, 1)$ [$\uparrow$ = exponentiation]
(1) – (6) are still identities of this structure; further identities are:

7 $1^x = 1$

8 $x^1 = x$

9 $x^{y+z} = x^y \cdot x^z$

10 $(x \cdot y)^z = x^z \cdot y^z$

11 $(x^y)^z = x^{y \cdot z}$.

(1–11) are called the High School Identities (HSI)

Tarski's HSI Problem (1960s)

Do the laws HSI form a basis for the identities of $(\mathbb{N}, +, \cdot, \uparrow, 1)$?

Answer is NO! (A. Wilkie, 1980)

The identities of $(\mathbb{N}, +, \cdot, \uparrow, 1)$ do not admit a finite basis. (R.Gurevič, 1990)

Definition

An algebraic structure is finitely based if there is a finite basis for its identities, otherwise it is non-finitely based (NFB). The finite basis problem (FBP) for a structure $\mathfrak{M}$ asks if that structure is finitely based or not.

We study the FBP for semigroups and involutory semigroups; an involutory semigroup is a semigroup S endowed with a unary operation * satisfying the identities

$$(xy)^* = y^*x^* \text{ and } (x^*)^* = x.$$

Intensely studied for **finite** semigroups.

Sufficient condition for an infinite semigroup / involutory semigroups to be NFB:

Theorem (ACHLV,2015)

A semigroup / involutory semigroup S is NFB provided that

1. $S \in \mathbf{Com} \textcircled{m} \mathbf{Fin}$
2. *S does not satisfy any identity of the form $Z_n = W$.*

Z_n is the nth Zimin word defined by $Z_1 = x_1$, $Z_{n+1} = Z_n x_{n+1} Z_n$.

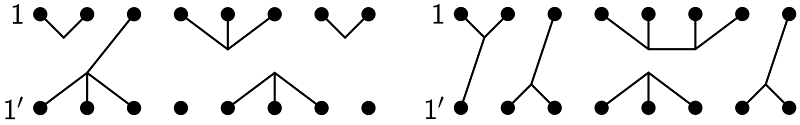
Choose and fix $n \in \mathbb{N}$.

Definition

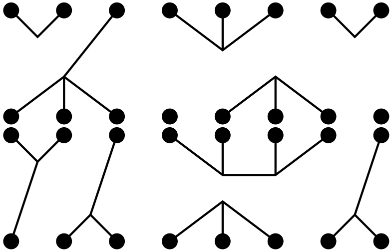
$P_n =$ all set partitions of $\{1, \ldots, n, 1', \ldots, n'\}$.

Subject to *composition of diagrams* this set becomes a monoid, the partition monoid P_n.

Composition of diagrams:



Composition of diagrams



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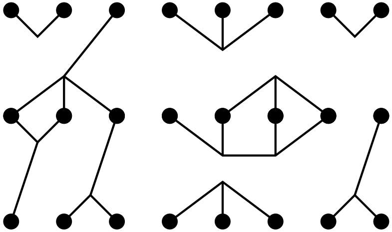

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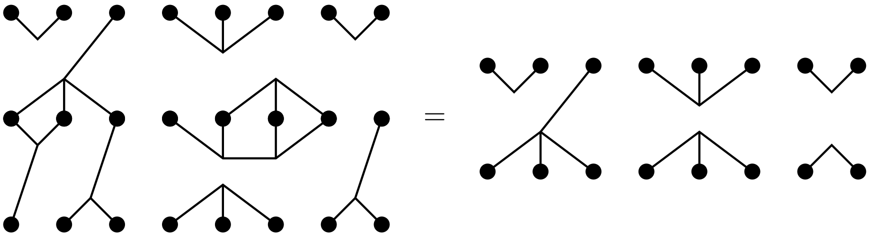


Definition

P_n = all set partitions of $\{1, \ldots, n, 1', \ldots, n'\}$.

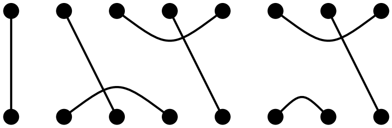
Subject to *composition of diagrams* this becomes a monoid, the partition monoid P_n.

Some prominent submonoids:

- *Brauer monoid* B_n
- *Jones monoid* = *Temperley–Lieb monoid* J_n
- *annular monoid* A_n

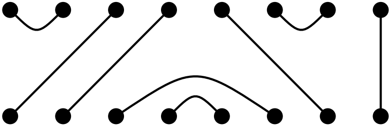
Brauer diagrams

all blocks have size two:



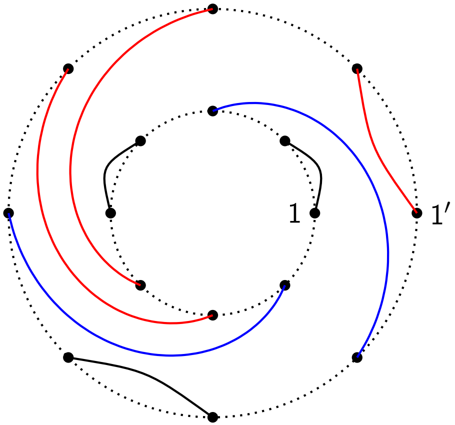
Temperley–Lieb diagrams

Brauer diagrams drawn without crossing lines within a rectangle:



Annular diagrams

Brauer diagrams drawn without crossings within an annulus:



Definition

For two diagrams $\alpha, \beta \in P_n$ denote by $\ell(\alpha, \beta)$ the number of floating blocks arising through the composition of α and β.

Definition (deformed partition monoid)

$$\mathcal{P}_n := P_n \times \mathbb{N}_0$$

endowed with the binary operation

$$(\alpha, k)(\beta, m) = (\alpha\beta, k + m + \ell(\alpha, \beta)).$$

Likewise:

- $\mathcal{B}_n = B_n \times \mathbb{N}_0$: *wire monoid*
- $\mathcal{J}_n = J_n \times \mathbb{N}_0$: *Kauffman monoid*
- $\mathcal{A}_n = A_n \times \mathbb{N}_0$ *deformed annular monoid*

The mapping $\mathcal{X}_n \twoheadrightarrow X_n$, $(\alpha, k) \mapsto \alpha$ is a surjective morphism for every $X \in \{P, B, J, A\}$. For any $(\alpha, k), (\alpha, m) \in \mathcal{X}_n$:

$$(\alpha, k)(\alpha, m) = (\alpha^2, k + m + \ell(\alpha, \alpha)) = (\alpha, m)(\alpha, k).$$

It follows that $\mathcal{X}_n \in \mathbf{Com} \circledm X_n$ for every $\mathcal{X} \in \{\mathcal{P}, \mathcal{B}, \mathcal{J}, \mathcal{A}\}$.

Theorem

1. $\mathcal{P}_n$ *is NFB iff* $n \geq 2$
2. $\mathcal{B}_n$, $\mathcal{J}_n$, $\mathcal{A}_n$ *are NFB iff* $n \geq 3$

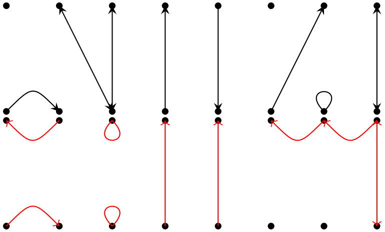
Choose and fix $n \in \mathbb{N}$.

Definition

MM_n = the set of all binary relations on $\{1, \ldots, n, 1', \ldots, n'\}$.

Subject to appropriate composition, this set of *partitioned binary relations* becomes a monoid MM_n (defined by Martin–Mazorchuk).

Composition of partitioned binary relations:



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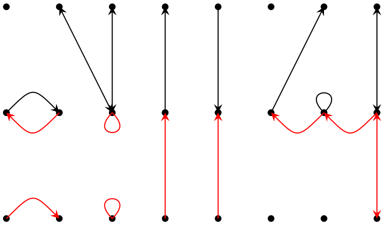

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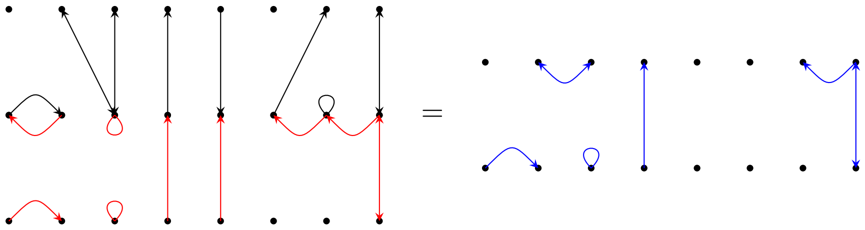

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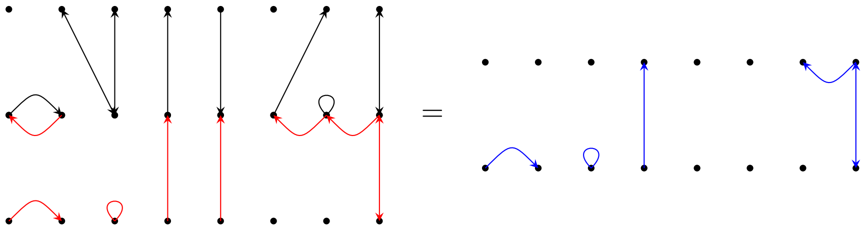

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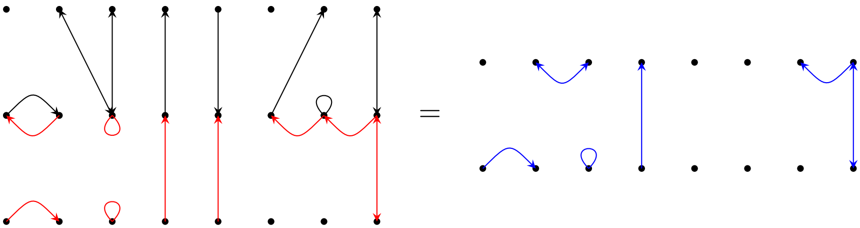
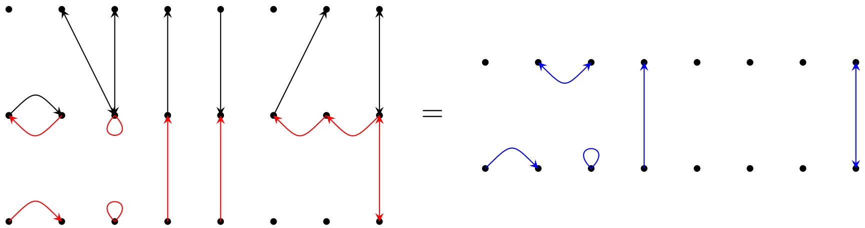


Composition of partitioned binary relations:



One can define the notion of floating block (= *"frothy cycle"*) and set, for $\alpha, \beta \in MM_n$:
$\ell(\alpha, \beta)$ = number of frothy cycles arising through the composition of α and β to obtain a deformed version of MM_n:

Definition (deformed monoid of partitioned binary relations)

$$\mathcal{MM}_n := MM_n \times \mathbb{N}_0$$

endowed with

$$(\alpha, k)(\beta, m) = (\alpha\beta, k + m + \ell(\alpha, \beta)).$$

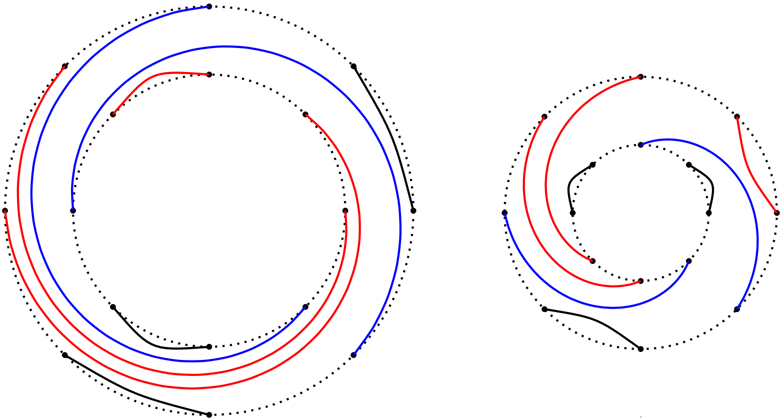
Again:

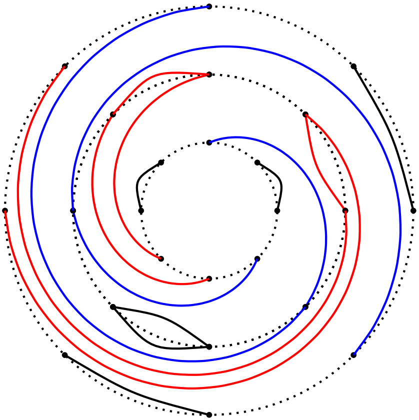
$$\mathcal{MM}_n \in \mathbf{Com} \circledcirc MM_n.$$

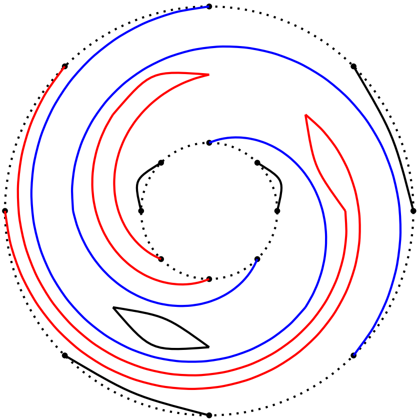
Theorem

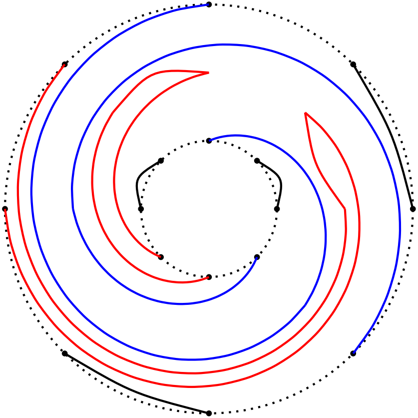
$\mathcal{MM}_n$ is NFB iff $n \geq 1$.

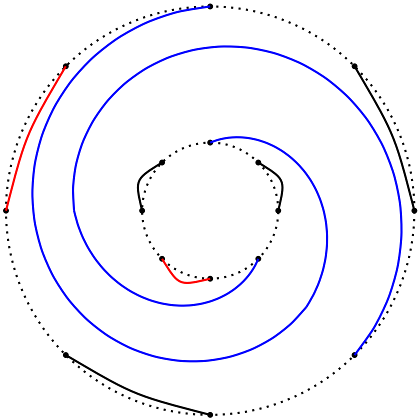
Examples of members of the annular monoid A_8:

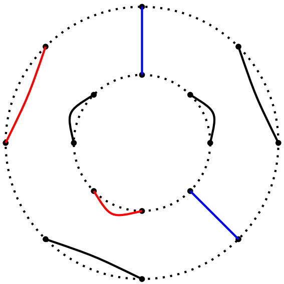










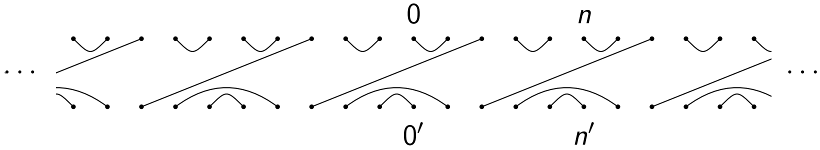


Let $\mathbb{Z}'$ be a disjoint copy of $\mathbb{Z}$

Definition (Affine Temperley–Lieb diagram of degree n)

this is a partition α of $\mathbb{Z} \cup \mathbb{Z}'$ such that

1. all blocks have size 2
2. for all $i, j \in \mathbb{Z} \cup \mathbb{Z}'$: $\{i, j\} \in \alpha \Leftrightarrow \{i + n, j + n\} \in \alpha$
3. the blocks can be drawn as non-crossing lines in a bi-infinite strip



Definition (Affine Temperley–Lieb monoid)

ATL_n = all affine Temperley-Lieb diagrams on degree n subject to composition of diagrams

Definition (Deformed affine Temperley-Lieb monoid)

$$\mathcal{ATL}_n = ATL_n \times \mathbb{N}_0 \times \mathbb{N}_0$$

subject to adequate multiplication.

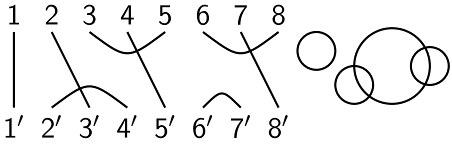
Fact:

$$ATL_n, \mathcal{ATL}_n \in \mathbf{Com} \text{ ⓜ } A_n.$$

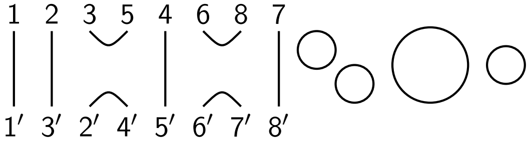
Theorem

1. *ATL_n is NFB iff $n \geq 3$*
2. *$\mathcal{ATL}_n$ is NFB iff $n \geq 3$.*

Example of 1-cobordism of degree 8:



is the same as this:



The monoid of 1-cobordisms of degree n coincides with the wire monoid $\mathcal{B}_n$:

$$1Cob_n = \mathcal{B}_n.$$

Definition (2-cobordism of degree n)

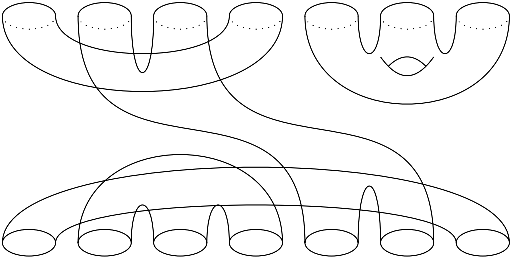
A 2-cobordism of degree n is a compact 2-dimensional manifold having $2n$ boundary components marked by $1, 2, \ldots, n, 1', \ldots, n'$.

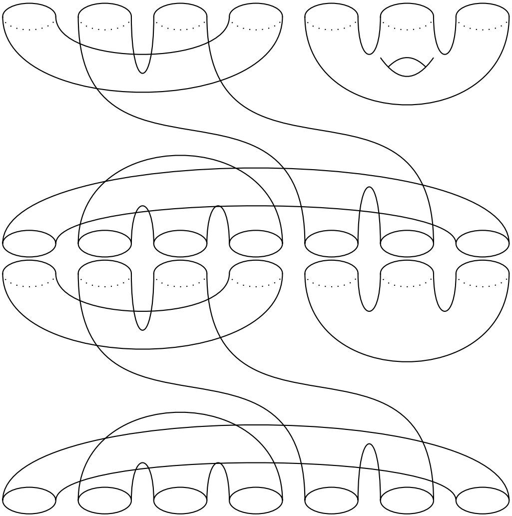
Definition (Monoid of 2-cobordisms of degree n)

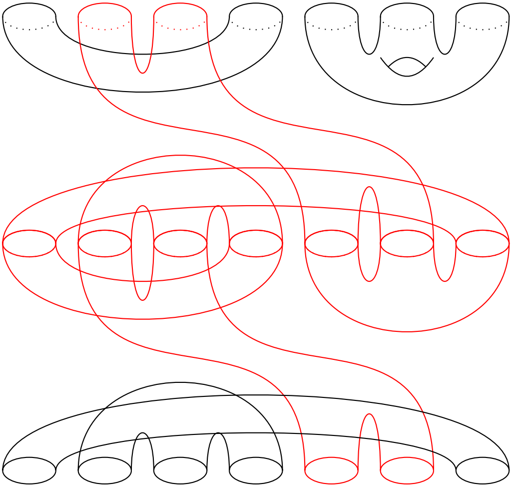
The composition of two 2-cobordisms is

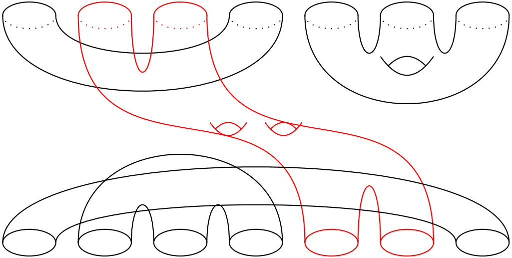
1. by disjoint union of the components without boundary
2. by concatenation of the components with boundary (as in the partition monoid)

Composition of cobordisms









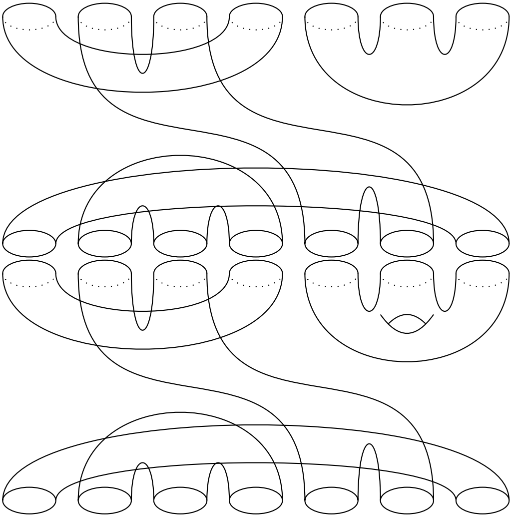
every 2-cobordism of degree n is uniquely determined by a triple (α, g, w) where:

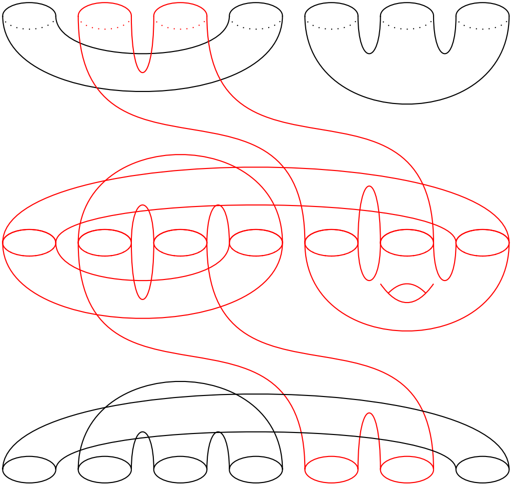
1. $\alpha \in P_n$ (partition induced on the boundary components)
2. $g : \alpha \to \mathbb{N}_0$ (genus of the components with boundary)
3. $w = \sum n_i x_i$ is a member of the free commutative monoid on $\{x_0, x_1, \dots\}$ (indicating n_i "floating" components of genus i, for every i)

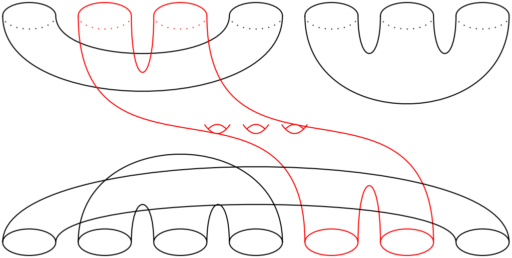
The mapping $2Cob_n \to P_n$, $(\alpha, g, w) \mapsto \alpha$ is a morphism.

Fact:

$$2Cob_n \notin \mathbf{Com} \circledcirc{\mathrm{m}} P_n.$$







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The mapping $2Cob_n \to P_n$, $(\alpha, g, w) \mapsto \alpha$ is a morphism.

Theorem

$$2Cob_n \in \mathrm{var}(\mathbf{CS}(\mathbf{Ab})) \circledcirc P_n.$$

Theorem

$2Cob_n$ is NFB iff $n \geq 1$.

Involutions

There are two involutions on P_n (also on MM_n):

1. the reflection * induced by the permutation $i \leftrightarrow i'$ for all i
2. the rotation $^\rho$ induced by the permutation $1 \leftrightarrow n', 2 \leftrightarrow (n-1)', \ldots, n \leftrightarrow 1'$.

These involutions can be canonically extended to all deformed and related monoids mentioned in the talk.

All results stay true for both versions of involutory semigroups.

$$2^2 \cdot 3 \cdot 5$$

Dear Gracinda, dear Jorge, all the best for the years to come!

Thanks!