Identifying The Structure of The Relatively Free Pro-**BSS** Forest Algebras

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Free Forest Algebra

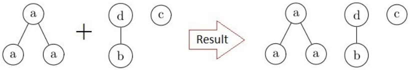


Figure: Addition of forests

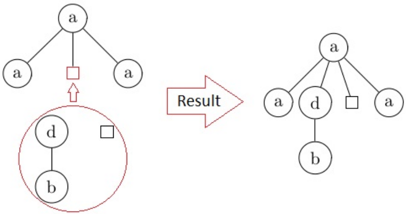


Figure: Product of contexts

Two More Operations

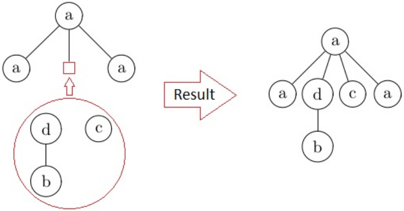


Figure: Action of a context on a forest

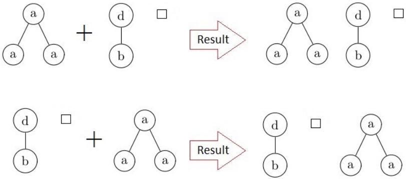


Figure: Addition of a context with a forest

Forest Algebra

Definition

A **forest algebra** S consists of a pair (H, V) of distinct monoids, subject to some additional requirements, which we describe below.

We write the operation in V, the vertical monoid, multiplicatively and the operation in H, the horizontal monoid, additively, although H is not assumed to be commutative. We accordingly denote the identity of V by $\Box$ and that of H by 0.

We require that V acts on the left of H. That is, there is a map

$$(v, h) \in V \times H \mapsto vh \in H$$

such that $w(vh) = (wv)h$, for every $h \in H$ and every $v, w \in V$. We also require that this action be **monoidal**, that is, $\Box.h = h$, for every $h \in H$, and that it be **faithful**, that is, if $vh = wh$, for every $h \in H$ then $v = w$.

Forest Algebra

Definition (...)

We further require that for every $h \in H$ and $v \in V$, V contains elements $h + v$ and $v + h$ such that for every $x \in S$,

$$(v + h)x = vx + h \quad \text{and} \quad (h + v)x = h + vx,$$

where vx is given by the action of v on x if x is a forest and by composition (multiplication) if x is a context.

Definition

Let (H_1, V_1) and (H_2, V_2) be algebras that satisfy the equational axioms of forest algebras. A **forest algebra homomorphism** $\alpha : (H_1, V_1) \to (H_2, V_2)$ is a pair (γ, δ) of monoid homomorphisms

$$\begin{aligned} \gamma : \quad & H_1 \to H_2, \\ \delta : \quad & V_1 \to V_2 \end{aligned}$$

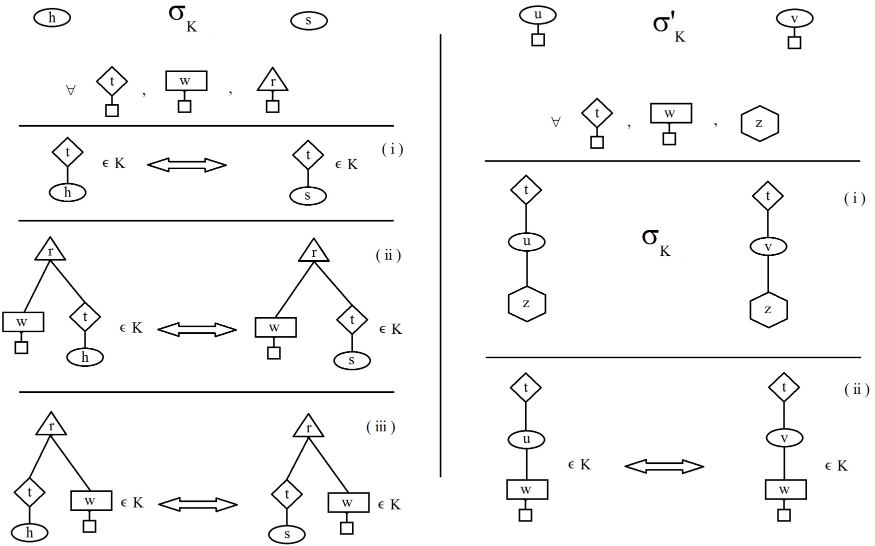
such that, for every $h \in H$ and every $v \in V$,

$$\gamma(vh) = \delta(v)\gamma(h) \qquad \text{and} \qquad \begin{cases} \delta(h+v) = \gamma(h) + \delta(v) \\ \delta(v+h) = \delta(v) + \gamma(h). \end{cases}$$

Fact

The image of a forest algebra homomorphism may not be a forest subalgebra and the pre-image of a forest subalgebra under a forest algebra homomorphism may not be a forest subalgebra.

Syntactic Congruence



Syntactic Congruence

Definition

Let $S = (H, V)$ be a forest algebra and K a subset of S. We may define on S a relation $\backsim_K = (\sigma_K, \sigma'_K)$, the so-called **syntactic congruence** of K, as follows:

- for $h_1, h_2 \in H$, $h_1 \; \sigma_K \; h_2$ if for all $t, w, r \in V$:
  - I. $th_1 \in K \iff th_2 \in K$;
  - II. $t(rh_1 + w) \in K \iff t(rh_2 + w) \in K$;
  - III. $t(w + rh_1) \in K \iff t(w + rh_2) \in K$.
- for $u, v \in V$, $u \; \sigma'_K \; v$ if for all $t, w \in V$ and $h \in H$:
  - I. $tuh \; \sigma_K \; tvh$;
  - II. $tuw \in K \iff tvw \in K$.

Lemma

For a forest algebra S and a subset K of S, the equivalence relations σ_K and σ'_K are congruences with respect to the basic operations of S.

The syntactic (forest) algebra for K is the quotient of S with respect to the equivalence $\backsim_K$, where the horizontal semigroup H_K consists of equivalence classes σ_K of forests in S, while the vertical semigroup V_K consists of equivalence classes σ'_K of contexts in S.

Definition

A subset $K = (H_K, V_K)$ of a forest algebra $S = (H, V)$ is called **inverse zero action subset** if,

$$H_K \subseteq H \qquad \text{and} \qquad V_K = \{v \in V \mid v * 0 \in H_K\}.$$

Proposition

Let $S = (H_S, V_S)$ be a forest algebra and let K be either a subset of H_S or inverse zero action subset of S. Then the quotient $S/\backsim_K$ is a forest algebra.

Pseudovariety

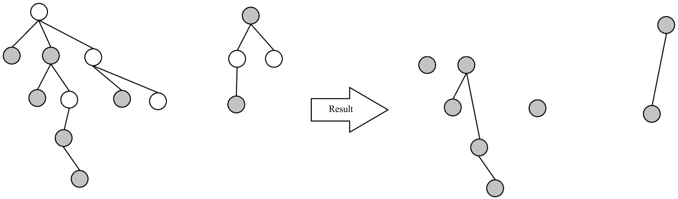
Definition

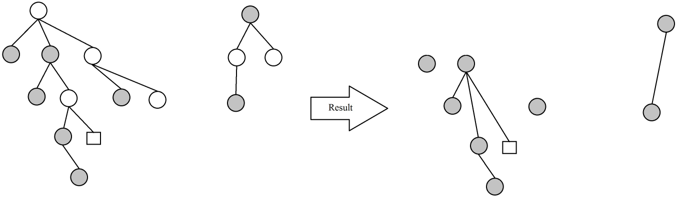
A nonempty class $\mathbf{V}$ of finite forest algebras is called a **pseudovariety** if the following conditions hold:

(i) if $S \in \mathbf{V}$ and B is a forest subalgebra of S, then $B \in \mathbf{V}$;

(ii) if $S \in \mathbf{V}$ and $S \to B$ is an onto forest algebra homomorphism, then $B \in \mathbf{V}$;

(iii) $\mathbf{V}$ is closed under **finite** direct products.


Result


Result

Definition

A **piece** of an element of the free forest algebra is obtained by removing some nodes from it while preserving the forest order and the ancestor relationship.

A forest language L over A is called **piecewise testable** if there exists $n \geq 0$ such that membership of t in L is determined by the set of pieces of t of size n or less. The size of a piece is the size of the forest, i.e. the number of nodes.

The pseudovariety **BSS** of finite forest algebras is generated by all syntactic forest algebras of piecewise testable forest languages.

For two elements $u, v \in A^{\Delta}$ and a forest algebra B if for every forest algebra homomorphism

$$\varphi : A^{\Delta} \to B$$

the equality $\varphi(u) = \varphi(v)$ holds, then we say that B **satisfies the identity** $u = v$ and we write $B \models u = v$. For a pseudovariety of finite forest algebras $\mathbf{V}$, define:

$$r(u, v) = \min\{|B| \mid \ B \in \mathbf{V} \ \ \textit{and} \ \ B \not\models u = v\}$$

and

$$d(u, v) = 2^{-r(u,v)}$$

where we take $\min \emptyset = \infty$ and $2^{-\infty} = 0$.
The function d is a pseudo-ultrametric on A^{Δ}, the basic operations on A^{Δ} are uniformly continuous and (A^{Δ}, d) is totally bounded.

Fix a finite set A. For a pseudovariety $\mathbf{V}$ of finite forest algebras a *pro-$\mathbf{V}$ forest algebra* is defined to be a projective limit of a projective system of A-generated finite forest algebras in $\mathbf{V}$.

Theorem

A zero-dimensional and compact metric forest algebra is residually finite.

Theorem

Let $\mathbf{V}$ be a pseudovariety of finite forest algebras and A be a finite set. An A-generated compact forest algebra S is a pro-$\mathbf{V}$ forest algebra if and only if S is residually in $\mathbf{V}$ as a topological forest algebra.

The Hausdorff completion of the ultrametric space (A^{Δ}, d), denoted by $\overline{\Omega}_A\mathbf{V}$, is a forest algebra.

Corollary

The forest algebra $\overline{\Omega}_A\mathbf{V}$ is a pro-$\mathbf{V}$ forest algebra.

A **V**-pseudoidentity is a formal equality $u = v$ with $u, v \in \overline{\Omega}_A \mathbf{V}$ for some finite set A. And for $S \in \mathbf{V}$, we write $S \models u = v$ if, for every continuous forest algebra homomorphism $\varphi : \overline{\Omega}_A \mathbf{V} \to S$, the equality $\varphi(u) = \varphi(v)$ holds.

Theorem (Analog of Reiterman Theorem)

Every pseudovariety of finite forest algebra is exactly classes of forest algebras definable by pseudoidentities.

ω-Algebras

Definition

An ω-*algebra* $\mathrm{B} = (\mathrm{H}, \mathrm{V})$ is a set with two types of elements endowed with six binary operations $+, +_1, +_2, ., *$, and $\mathrm{cm}(,)$ and two unary operations $\omega()$ on H and $()^\omega$ on V, such that the following conditions are satisfied:

1. equational axioms of forest algebras;
2. $\omega(0) = 0$;
3. $\mathrm{cm}(\square, h) = \omega(h)$;
4. $(\square)^\omega = \square$;
5. for every $h, s \in \mathrm{H}$, $(h +_1 \square +_2 s)^\omega = \omega(h) +_1 \square +_2 \omega(s)$;

The class of ω-algebras is closed under direct products and subalgebras. So, all the free ω-algebras exist.

Theorem

Every free ω-algebra $\mathcal{A}$ is a forest algebra.

We distinguished all kinds of non-trivial additively irreducible and non-trivial multiplicatively irreducible elements of the free ω-algebras:

Lemma

Let v and u be p-contexts and h be a p-forest in the free ω-algebra $\mathcal{A}$ with $v \notin (\mathsf{H} + \Box + \mathsf{H})$. Let $a \in A$. Then

$$v^{\omega} * h \ , \quad a\Box * h \ , \quad a\Box . u \ , \quad v^{\omega}.u \ , \quad \mathrm{cm}(v,h) \ , \quad \textit{and } \omega(h) \textit{ for } h \neq 0$$

are additively irreducible. And

$$s + \Box \ , \quad \Box + s \ , \quad v^{\omega} \ , \quad \textit{and } a\Box,$$

where s is a non-trivial additively irreducible p-forest, are multiplicatively irreducible.

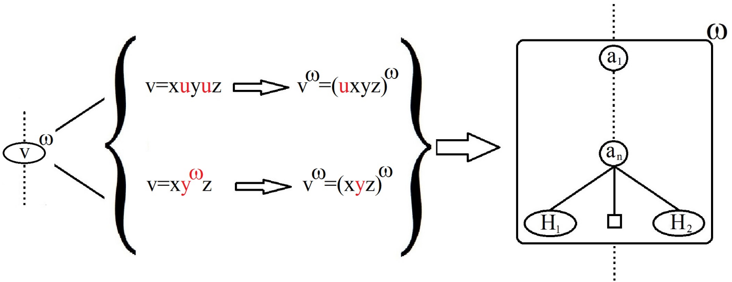
Identities

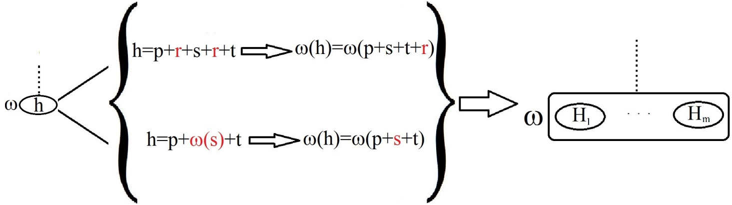
Considering the variety $\mathcal{V}$ of ω-algebras, defined by the set Σ consisting of the following identities, for context terms w, u, and v and forest terms h and s,

$$
\begin{array}{ll}
vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh, & (v^\omega)^\omega = v^\omega, \\
(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega, & v^\omega v = v^\omega = v v^\omega, \\
\mathrm{cm}(u, \mathrm{cm}(uv, h)) = \mathrm{cm}(uv, h), & \omega(\mathrm{cm}(u, h)) = \mathrm{cm}(u, h), \\
\mathrm{cm}(u + s, h) = \mathrm{cm}(u, h + s), &
\end{array}
$$

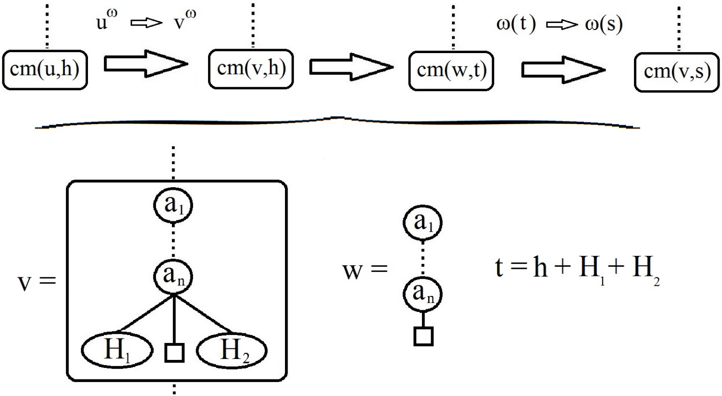
$$
\begin{aligned}
\mathrm{cm}(u, h) &= (u(\mathrm{cm}(u, h) + \square))^\omega \mathrm{cm}(u, h) \\
&= (u(\square + \mathrm{cm}(u, h)))^\omega \mathrm{cm}(u, h), \\
&= \omega(uh) + \mathrm{cm}(u, h) = \mathrm{cm}(u, h) + \omega(uh), \\
&= \mathrm{cm}(u^\omega, h) = \mathrm{cm}(u, \omega(h)).
\end{aligned}
$$

Ordered Form





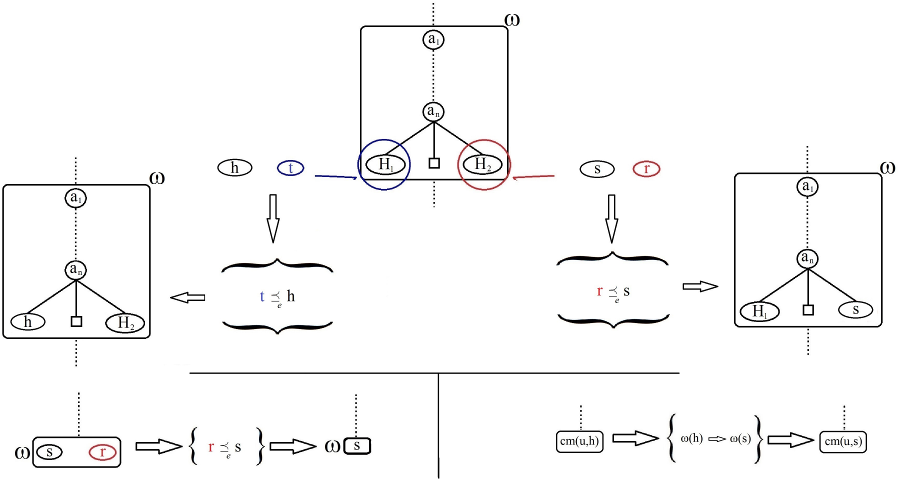
Ordered Form



$x \preceq_e y$: (Extended Piece)

- $y = f(y_1, \ldots, y_n)$ and $x = f(x_1, \ldots, x_n)$ where f is a composition of basic operation of forest algebras and for every i, y_i is either $a\Box$, u^ω, $\omega(h)$, or $\mathrm{cm}(u, h)$;
- if $y_i = a\Box$ then x_i is either $a\Box$ or $\Box$;
- if $y_i = \omega(h)$ then $x_i = t_1 + \cdots + t_m$ as the sum of non-trivial additively irreducible p-forests such that for $t_j \neq \omega(s)$, $t_j \preceq_e h$ and for $t_j = \omega(s)$, $s \preceq_e \omega(h)$;
- if $y_i = u^\omega (= (b_1\Box \cdots b_k\Box(H_1 + \Box + H_2))^\omega)$ then $x_i = \prod v_j$ as the product of non-trivial multiplicatively irreducible p-contexts such that:
  - $\mathrm{labels}(\mathrm{nerve}(x_i)) \subseteq \{b_1, \ldots, b_k\}$;
  - for every factor $v_j = t + \Box$, $t \preceq_e \omega(H_1)$ and for every factor $v_j = \Box + s$, $s \preceq_e \omega(H_2)$;
  - for every factor $v_j = v^\omega$, $v \preceq_e u^\omega$.

- if $y_i = cm(u, h)$ with $u = b_1\Box \cdots b_k\Box$ then $x_i = t_1 + \cdots + t_m$ as the sum of non-trivial additively irreducible p-forests such that for every j:
  - If t_j is different from $\mathrm{cm}(q, p)$ and $\omega(s)$, then one of the following holds:
    - $t_j \preceq_e \omega(h)$.
    - $t_j = a\Box z$ such that $a \in \{b_1, \ldots, b_k\}$ and $z \preceq_e \mathrm{cm}(u, h)$.
    - $t_j = v^\omega z$ with $v^\omega = (a_1\Box \cdots a_{k'}\Box(H_1 + \Box + H_2))^\omega$ such that $\{a_1, \ldots, a_{k'}\} \subseteq \{b_1, \ldots, b_k\}$, $H_1 + H_2 + z \preceq_e \mathrm{cm}(u, h)$.
  - If $t_j = \omega(s)$, then $s \preceq_e \mathrm{cm}(u, h)$.
  - If $t_j = \mathrm{cm}(q, p)$, then $\mathrm{labels}(\mathrm{nerve}(q)) \subseteq \{b_1, \ldots, b_k\}$ and $p \preceq_e \mathrm{cm}(u, h)$.


ω
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a_n
H_1
H_2
h
t
s
r
$t \preceq_e h$
$r \preceq_e s$
ω
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a_n
h
H_2
ω
a_1
a_n
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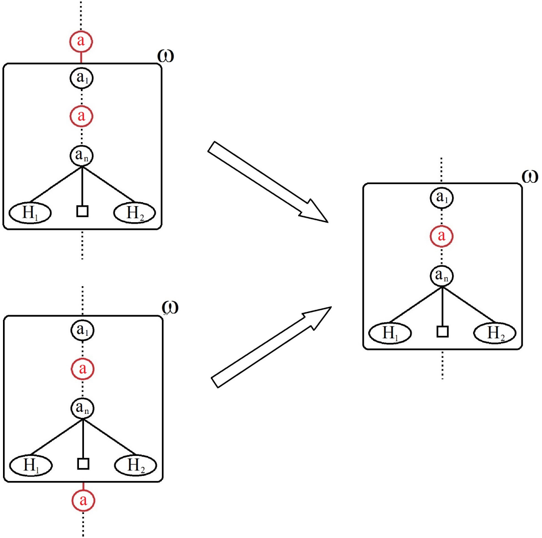
s
ω s r
$r \preceq_e s$
ω s
cm(u,h)
ω(h) ⇒ ω(s)
cm(u,s)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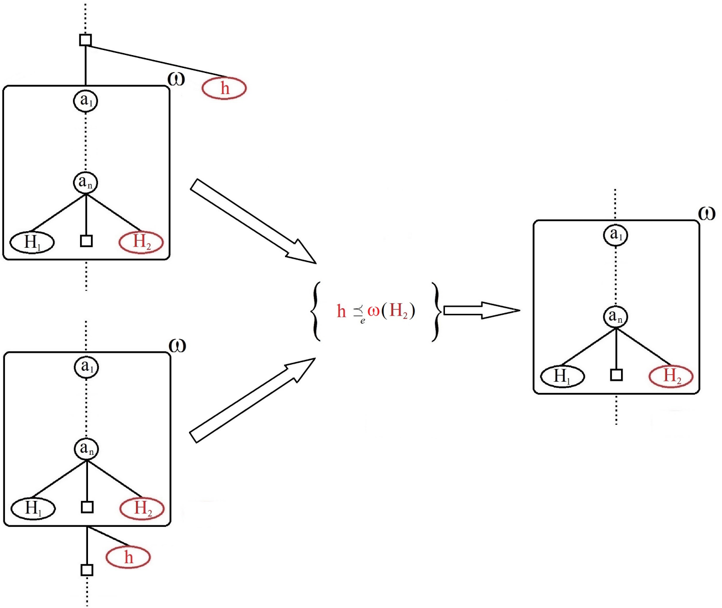
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$h \preceq_e \omega(H_2)$
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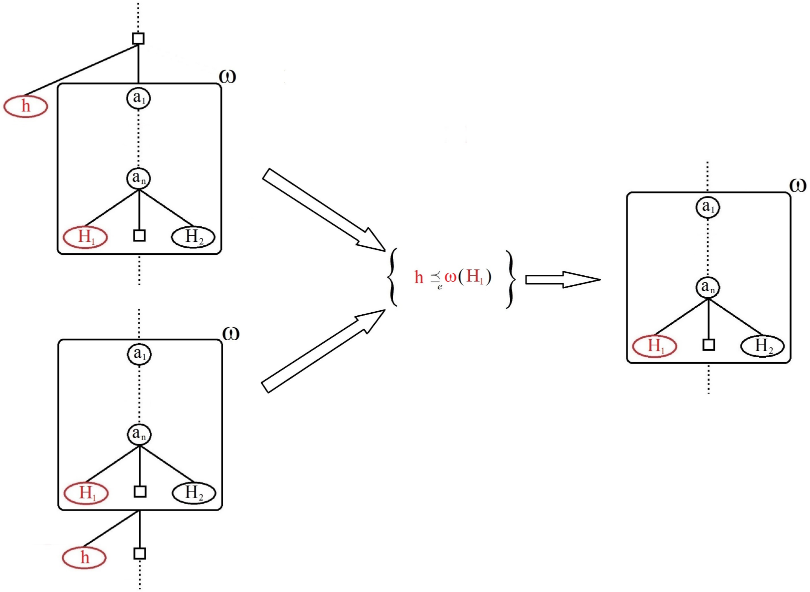

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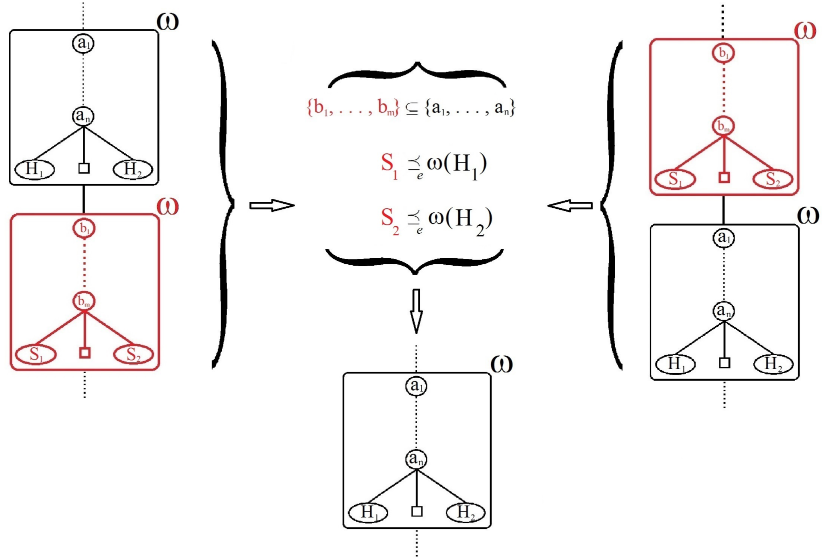
ω
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H2
b1
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S1
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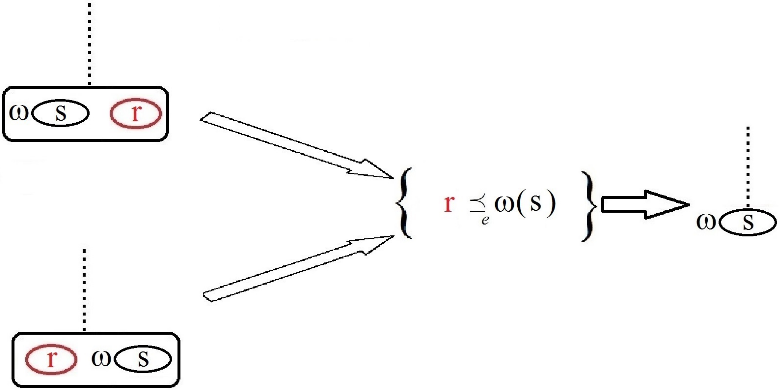
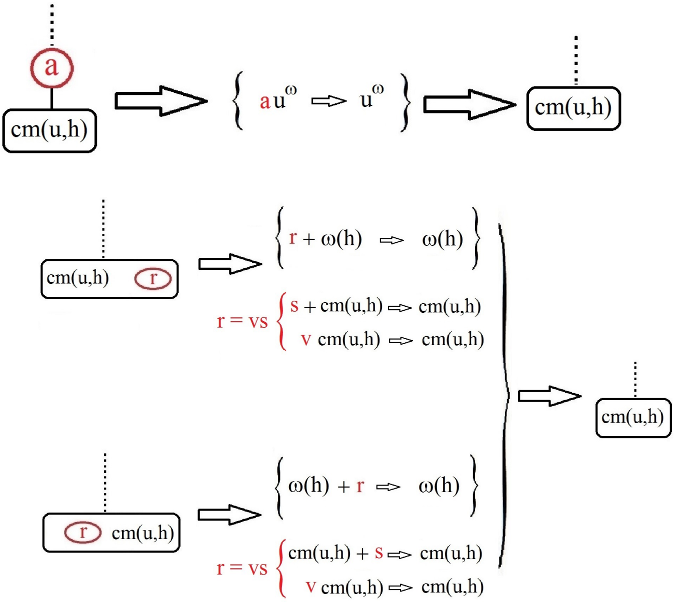
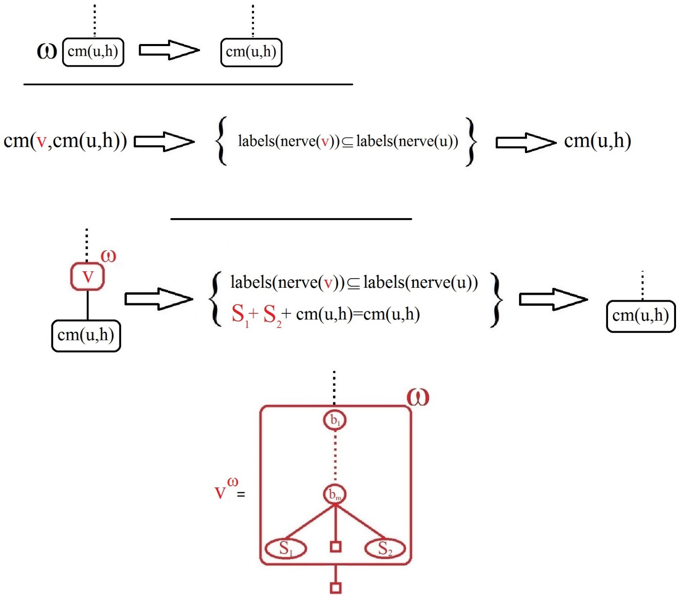
{b1, . . . , bm} ⊆ {a1, . . . , an}
S1 ⪯e ω(H1)
S2 ⪯e ω(H2)


ω s r
r ω s
r $\preceq_e$ ω(s)
ω s


a
cm(u,h)
a u^ω ⇨ u^ω
cm(u,h)
cm(u,h) r
r + ω(h) ⇨ ω(h)
r = vs
s + cm(u,h) ⇨ cm(u,h)
v cm(u,h) ⇨ cm(u,h)
cm(u,h)
r cm(u,h)
ω(h) + r ⇨ ω(h)
r = vs
cm(u,h) + s ⇨ cm(u,h)
v cm(u,h) ⇨ cm(u,h)


ω cm(u,h)
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labels(nerve(v))⊆labels(nerve(u))
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labels(nerve(v))⊆labels(nerve(u))
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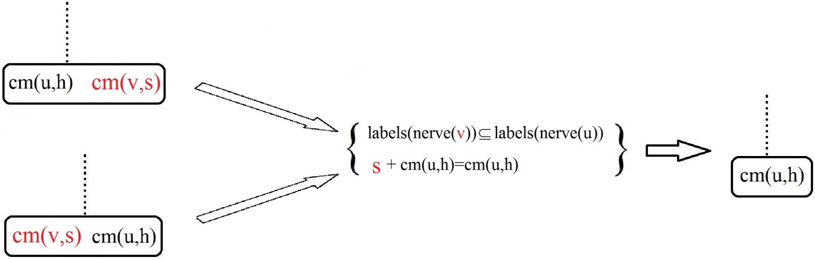
cm(u,h) cm(v,s)
cm(v,s) cm(u,h)
labels(nerve(v))⊆labels(nerve(u))
s + cm(u,h)=cm(u,h)
cm(u,h)

We then have a system of reduction rules which is noetherian and confluent. This implies that for elements t_1 and t_2 in $\mathcal{A}$ with $t_1 \sim_\Sigma t_2$ if we apply the reduction rules on t_1 and t_2, then the results are the same. The variety $\mathcal{V}$ certainly contains **BSS**. Denoting by $F_A\mathcal{V}$ the $\mathcal{V}$-free algebra on A, we then have an ω-algebra homomorphism

$$\varphi : F_A\mathcal{V} = (H_1, V_1) \to \overline{\Omega}_A\mathbf{BSS} = (H_2, V_2)$$

such that $x_i \mapsto x_i$ $(i = 1, \ldots, n)$.
If two p-contexts or p-forests have the same canonical form, then in $F_A\mathcal{V}$ they are equal and so they have the same image by φ. Therefore, their image by φ have the same set of pieces.

Theorem

The ω-algebra homomorphism φ is bijective.

Example

Let $T = \{0, 1\}$, $H = \mathbb{N}$ and $V = H \times T$. For elements (s_1, t_1) and (s_2, t_2) in V and elements s in H, define:

$$(s_1, t_1).(s_2, t_2) = \begin{cases} (s_1 + s_2, t_2) & ,(s_2, t_2) \neq (0, 0) \\ (s_1, t_1) & ,(s_2, t_2) = (0, 0), \end{cases}$$

$s +' (s_1, t_1) = (s + s_1, t_1)$, $(s_1, t_1) +' s = (s_1 + s, t_1)$, and

$$(s_1, t_1) * s = \begin{cases} s_1 + s & ,s \neq 0 \\ s_1 & ,s = 0 \text{ and } t_1 = 0 \\ s_1 + 1 & ,s = 0 \text{ and } t_1 = 1. \end{cases}$$

Then (H, V) satisfies the equational axioms of forest algebras. By the universal property of the free forest algebra A^Δ, there is a unique forest algebra homomorphism $\#_{\text{Leaves}} : A^\Delta \to (H, V)$ such that $\#_{\text{Leaves}}(a\Box) = (0, 1)$.

Theorem

The variety of type τ generated by **BSS** *is defined by the identities*

$$v^\omega v = v^\omega = v v^\omega$$
$$(uv)^\omega = (vu)^\omega = (u^\omega v^\omega)^\omega$$
$$(v^\omega)^\omega = v^\omega$$

$$vh + \omega(vuh) = \omega(vuh) = \omega(vuh) + vh$$

$$\mathrm{cm}(u,h) = \omega(uh) + \mathrm{cm}(u,h) = \mathrm{cm}(u,h) + \omega(uh)$$
$$\omega(\mathrm{cm}(u,h)) = \mathrm{cm}(u,h)$$
$$\mathrm{cm}(u, \mathrm{cm}(uv,h)) = \mathrm{cm}(uv,h)$$
$$(u(\mathrm{cm}(u,h) + \square))^\omega \mathrm{cm}(u,h) = \mathrm{cm}(u,h) = (u(\square + \mathrm{cm}(u,h)))^\omega \mathrm{cm}(u,h)$$
$$\mathrm{cm}(u, h+s) = \mathrm{cm}(u+s, h)$$
$$\mathrm{cm}(u,h) = \mathrm{cm}(u^\omega, h) = \mathrm{cm}(u, \omega(h))$$

(...)

and $\overline{\Omega}_A$**BSS** is the free object on A in this variety. Two terms in the variables from A coincide in $\overline{\Omega}_A$**BSS** if and only if they have the same canonical form with respect to the reduction rules. In particular, the word problem for $\overline{\Omega}_A$**BSS** is decidable.

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