ESTUDOS
DE MATEMÁTICA

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em homenagem ao

Prof. A. ALMEIDA COSTA

LISBOA
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BREVE NOTA SOBRE A ACTIVIDADE CIENTÍFCA DO PROF. A. ALMEIDA COSTA

O Prof. António Almeida Costa, de seu nome completo, nasceu em Celorico da Beira, em 25 de Maio de 1903. Fez o curso dos liceus no Liceu da Guarda e concluiu-o com a classificação de 19 valores. Em 1924, licenciou-se em Ciências Matemáticas na Faculdade de Ciências do Porto, também com 19 valores. Aí recebeu os prémios de Gomes Teixeira e de Gomes Ribeiro. Entre 1924 e 1952 desempenhou, na mesma Faculdade, as funções sucessivas de assistente, de professor auxiliar (1933), de professor extraordinário (cerca de 1938) e de professor catedrático (1950, Mecânica Celeste), sempre dentro do grupo das Matemáticas Aplicadas. De 1937 a 1939 fez estudos de Física Teórica na Universidade de Berlim, participando *mit sehr gutem Erfolg* em vários cursos avançados. Em resultado desses estudos pronunciou em Lisboa, em 1945, a convite da Sociedade Portuguesa de Matemática, algumas conferências sobre *Álgebras em Quântica*.

A convite da Junta de Investigação de Matemática dirigiu as publicações de Álgebra moderna da mesma Junta.

Em 1952, aceitou o convite que lhe foi dirigido pela Faculdade de Ciências de Lisboa, no sentido de ocupar o lugar de professor catedrático de Álgebra. Desde então, até à data do seu jubileu (1973), ensinou em Lisboa as disciplinas de Geometria Descritiva, Geometria Superior e Física Matemática, juntamente com várias disciplinas de Algebra. A Faculdade de Ciências de Lisboa outorgou-lhe em 1961 o título de Doutor.

Do que foi a acção renovadora do Prof. Almeida Costa no campo do ensino da Álgebra em Portugal, da importância e tenacidade dessa acção, que se exerceu tanto nos cursos de licenciatura, como nos trabalhos de post-graduados, dão testemunho as linhas do Prof. Dias Agudo que adiante se publicam, as quais contêm o essencial das palavras proferidas pelo então subdirector (depois director) da Faculdade de Ciências de Lisboa, na ocasião da última lição do Prof. Almeida Costa.

Em Lisboa desempenhou o Prof. Almeida Costa funções várias, tais como: Director do Centro de Matemáticas Aplicadas ao Estudo da Energia Nuclear, da superintendência da Comissão de Estudos da Energia Nuclear; membro do Conselho de Investigação Científica do Instituto de Alta Cultura; membro do Conselho de Ciência da Fundação Gulbenkian; vogal da Comissão anexa à 1.ª secção da Junta Nacional da Educação; director da Faculdade de Ciências (Março 1972 — Maio 1973); e director de projectos de investigação do Instituto de Alta Cultura.

O Prof. Almeida Costa é membro efectivo da Academia das Ciências de Lisboa e sócio honorário da Real Sociedade de Matemática Espanhola. Em 1973, foi agraciado com a Comenda de Santiago da Espada.

Participou em numerosos congressos científicos em Lisboa, Porto, Coimbra, Málaga, Madrid, Sevilha, Budapeste, Edimburgo e Estocolmo. Participou no Simpósio da Teoria dos Anéis organizado pela Sociedade Matemática da República Federal Alemã em Oberwolfach, em 1962. Fez, por várias vezes, conferências no Instituto Henri Poincaré, de Paris (Seminário Dubreil-Pisot), e tomou parte nas Jornadas de Algebra realizadas neste mesmo Seminário por ocasião do centenário da Sociedade Matemática de França.

As primeiras Jornadas Matemáticas Luso-Espanholas, que tiveram lugar em Lisboa, em 1972, deveram a sua realização à iniciativa do Prof. Almeida Costa. Também dirigiu a participação portuguesa nas segundas Jornadas Matemáticas peninsulares, realizadas em Madrid, em 1973.

Os trabalhos do Prof. Almeida Costa encontram-se dispersos por várias publicações: Anais da Faculdade de Ciências do Porto, Gazeta de Matemática de Lisboa, Actas da Associação Espanhola para o Progresso das Ciências, Revista da Faculdade de Ciências de Lisboa, Boletim e Memórias da Academia das Ciências de Lisboa, Publicationes Mathematicae de Debrecen (Hungria), Mathematische Zeitschrift, Seminário Dubreil-Pisot, etc. Os primeiros desses trabalhos versam questões de Análise Matemática, de Geometria e de Mecânica Racional. Nos últimos trinta e dois anos, porém, pode dizer-se que a actividade investigadora do Prof. Almeida Costa foi inteiramente dedicada à Álgebra.

Problemas de dimensão de módulos sobre certos tipos de anéis, questões relativas a extensões inseparáveis de corpos, questões gerais

da teoria dos anéis, ocupam o Prof. Almeida Costa em muitos dos seus artigos.

De 1948 a 1951, deu contribuições importantes para a teoria dos anéis de endomorfismos. Usando os conceitos de ideal de contracção (dum módulo num submódulo) e de ideal aniquilador, dá teoremas inspirados em teoremas da teoria dos anéis e estabelece, de forma inesperada, alguns resultados respeitantes a módulos que satisfazem às duas condições de cadeia.

Em 1960, o Prof. Almeida Costa introduziu o conceito de anel-μ. Um dos signatários desta notícia estendeu esse conceito a semi-anéis, tendo-se o Prof. Almeida Costa ocupado posteriormente do assunto, em consequência do que o seu nome aparece ligado a uma caracterização dos mesmos.

O Prof. Almeida Costa ocupou-se ainda dos semi-anéis reticulados, contribuindo para o esclarecimento da sua estrutura no tocante a radicais, ideais mínimos, subsemi-anéis de divisão, etc.

Temos conhecimento de que o Prof. Almeida Costa pensa publicar outros trabalhos sobre semi-anéis, aliás em ligação com problemas de que se ocupou outro dos signatários.

Ao terminar esta pequena nota sobre a actividade do Prof. Almeida Costa, os signatários, seus antigos discípulos, e agora professores na Faculdade de Ciências de Lisboa, desejam reiterar-lhes a expressão da sua estima e os votos de uma vida longa e tão fecunda como a passada.

Maria Luísa Noronha Galvão
Professora extraordinária

Margarita Benito Ramalho
Professora auxiliar

João Cândido Furtado Coelho
Professor auxiliar

VINTE ANOS DEPOIS (*)

POR

F. R. DIAS AGUDO

A última lição do Professor Almeida Costa faz-me regressar um quarto de século atrás e levou-me a pensar no que era o curriculo escolar da licenciatura em matemática no meu tempo de estudante, em especial no que respeita à então chamada álgebra moderna. Um jovem licenciado em matemática de 1947 tinha ouvido falar em grupos de substituições, havia analisado o problema da resolubilidade algébrica, e pouco mais.

Já era finalista quando, numa série de colóquios realizados nesta Faculdade, comecei a familiarizar-me com a teoria dos grupos numa forma mais geral; entre a bibliografia aconselhada figurava uma obra de um autor português — um livro de um professor da Faculdade de Ciências do Porto; e foi assim que, nesse ano lectivo de 1946/47, contactei pela primeira vez com as obras de álgebra do Professor António Almeida Costa.

Depois de licenciado em matemática, andei pelo Instituto Superior de Agronomia e pelo Instituto Superior Técnico, ensinando e aprendendo, para, em 1951/52, regressar a esta escola — à minha Faculdade — como segundo assistente. Mais ou menos nessa altura era criada, no segundo ano da licenciatura em matemática, uma cadeira anual de Álgebra Superior, e convidado a transferir-se para Lisboa e a reger a parte não clássica o Doutor Almeida Costa.

As obrigações docentes de um assistente eram então bem mais pesadas do que agora; o tempo disponível para o estudo pessoal e a investigação bem mais reduzido; mas isso não impediu que o novo professor de Álgebra pusesse em marcha um seminário para o estudo dos assuntos que o apaixonavam, que conseguisse transmitir a alguns

(*) Da alocução proferida quando da última lição do Prof. Almeida Costa, em 25 de Maio de 1973.

jovens de então o seu entusiasmo pela álgebra «moderna»; e durante anos lectivos, pontualmente (uma vez por semana, pelo menos) suce-deram-se as exposições de seminário sobre *Teoria dos grupos*, *Teoria dos anéis e ideais não comutativos*, *Teoria dos corpos*. A sua volumosa obra «Sistemas hipercomplexos e representações» passou a tornar-se-nos familiar; e, talvez por isso, em Outubro de 1954, eramos nós próprios convidados, por indicação do Prof. Almeida Costa, para reger a cadeira de Física Matemática e nela ensinar *Representações de grupos*.

Tendo-me interessado até então pela teoria das matrizes, sua equação característica, valores próprios, tive ocasião, nessa disciplina, de contactar com os operadores lineares numa forma mais geral e de me entusiasmar pelas suas aplicações à análise, em particular às equações diferenciais. E quando, em 1956/57, me foi proporcionada uma bolsa para os Estados Unidos, acabei por optar pelo estudo dos operadores lineares em espaços de Hilbert. Pareceu-me mais tarde que o Professor Almeida Costa, que tanto se havia interessado pela concessão dessa bolsa, teria ficado magoado com esta minha passagem da álgebra para a análise. Mas eu sempre entendi o seu descontentamento como resultante da sua grande e justificada paixão pela álgebra. De modo nenhum a minha predilecção pela teoria dos operadores lineares em espaços de Hilbert significava menos apreço por quanto lhe ficara a dever pela preparação algébrica que então possuía; e, afinal, também ele contribuíra para me lançar na análise funcional ao indicar-me para reger a cadeira de Física Matemática.

Com este testemunho, com a recordação da minha própria vivência dos problemas, outro objectivo não tive do que tentar dar mais realce à actuação que o Prof. Almeida Costa teve nesta escola como grande impulsionador da álgebra. A sua dedicação à Faculdade, à sua primeira escola superior (o Doutor Almeida Costa foi caloiro da Faculdade de Ciências de Lisboa), os esforços que tem desenvolvido para a dotar de boas revistas e conseguir melhores instalações, tudo é bem conhecido. A sua actuação como director é dos nossos dias e já foi ou será apreciada por quem tem mais competência do que eu para o fazer. A sua actividade como algebrista não sofreu interrupções (basta referir os seus mais recentes volumes de Álgebra Geral) e não terminará com a jubilação. A obra que aqui começou a desenvolver há pouco mais de vinte anos continua, quer através das suas próprias lições (em cursos livres), quer por acção dos seus discípulos, que hoje já são professores.

TRABALHOS PUBLICADOS

pelo Prof. ALMEIDA COSTA

1. *Notas de Cálculo Vectorial*, 52 págs. Porto, 1931.
2. *Sobre a dinâmica dos sistemas holónomos*, 133 págs., Porto, 1932.
3. *Nota sobre o integral de* JACOBI *das equações de 1.ª ordem*. Anais Fac. Ci. Porto, **17** (1932), 3 págs..
4. *Nota sobre a integração dos sistemas canónicos*. Anais Fac. Ci. Porto, **18** (1933), 6 págs..
5. *Sobre a noção de reciprocidade em Cálculo Vectorial*. Anais Fac. Ci. Porto, **18** (1933), 4 págs..
6. *Elementos da Teoria dos Grupos*, 153 págs., 1942, Centro de Estudos Matemáticos Fac. Ci. Porto.
7. *Sobre os grupos abelianos*. Anais Fac. Ci. Porto, **27** (1942), 38 págs..
8. *Grupos abelianos e anéis e ideais não comutativos*, 173 págs., 1942, Centro de Estudos Matemáticos Fac. Ci. Porto.
9. *Elementos da Teoria dos Anéis*, 282 págs., 1943, Centro de Estudos Matemáticos Fac. Ci. Porto.
10. *Sobre Física Teórica*. «Gazeta de Matemática», Lisboa (1943), 3 págs..
11. *Sobre os anéis semi-primários*. Anais Fac. Ci. Porto, **29** (1944), 32 págs..
12. *Sobre os corpos comutativos*. Anais Fac. Ci. Porto, **31** (1946), 32 págs..
13. *Sistemas hipercomplexos e representações*, 518 págs., 1948, Centro de Estudos Matemáticos Fac. Ci. Porto.
14. *Sobre os endomorfismos dos módulos*. Anais Fac. Ci. Porto, **33** (1948), 28 págs..
15. *Sobre nilideais e ideais quási-regulares*. Anais Fac. Ci. Porto, **34**, (1949), 26 págs..
16. *Sobre a Teoria dos Anéis e Ideais não comutativos*, 50 págs, 1950, tomo I das Actas do XIII Congresso Luso-Espanhol para o Progresso das Ciências.
17. *Sobre anéis de endomorfismos*. «Gazeta de Matemática», Lisboa (1950), 5 págs..
18. *Über Kontraktions- und Vernichtungsideale in der allgemeinen Modulntheorie*. Rev. Fac. Ci. Lisboa, **1** (1951), 297-344.
19. *Sobre ideais de contracção e aniquiladores na teoria geral dos módulos*. Anais Fac. Ci. Porto, **35** (1951), 80 págs..

20. *Para a história dos domínios multiplicativos associativos*, 43 págs., **1951**, Madrid, Memórias do XXI Congreso da Associación Española para el Progreso de las Ciencias.
21. *Três lições sobre a teoria geral dos anéis, 1.ª lição: Radical-G. Anti-radical. Ideal regular máximo dum anel.* Anais Fac. Ci. Porto, **36** (1952), 19 págs..
22. *Três lições sobre a teoria geral dos anéis, 2.ª lição: Anéis primitivos.* Anais Fac. Ci. Porto, **36** (1952), 32 págs..
23. *Três lições sobre a teoria geral dos anéis, 3.ª lição: Somas subdirectas de anéis. Anéis semi-simples.* Anais Fac. Ci. Porto, **36** (1952), 27 págs..
24. *Über die unterdirekten Modulnsummen.* Rev. Fac. Ci. Lisboa, **2** (1952), 115-160.
25. *Três lições sobre a teoria geral dos anéis, aplicações e complementos, I (Somas subdirectas de módulos, módulos semi-simples, submódulo-G).* Anais Fac. Ci. Porto, **37** (1953), 42 págs..
26. *Matrizes constantes e periódicas.* Rev. Fac Ci. Lisboa, **3** (1954), 61-74.
27. *Curso de Álgebra abstracta*, 240 págs , 1954, Lisboa.
28. *On modules and rings with operators.* Rev. Fac. Ci. Lisboa, **4** (1954), 5-62.
29. *Três lições sobre a teoria geral dos anéis, aplicações e complementos, II (Módulos e anéis com operadores, anéis simples e álgebras simples).* Anais Fac. Ci. Porto, **37** (1954), 67 págs..
30. *Uber die nichtassoziativen Ringe, die halbeinfache Moduln sind.* Rev. Fac. Ci. Lisboa, **5** (1955), 75-102.
31. *Über die Fastgruppentheorie*, Rev. Fac. Ci. Lisboa, **5** (1956), 265-328.
32. *Sur les anneaux demi-premiers.* Rev. Fac. Ci. Lisboa, **7** (1958), 89-104.
33. *Sur le suprêmum d'une famille de relations d'équivalence.* Rev. Fac. Ci. Lisboa, **7** (1958), 121-132.
34. *Elementos de álgebra linear e de geometria linear*, 272 págs., 1958, Lisboa.
35. *μ-systèmes et π-systèmes d'idéaux.* Rev. Fac. Ci. Lisboa, **7** (1959), 235-243.
36. *μ-anneaux et μ_0-anneaux.* Rev. Fac. Ci. Lisboa, **8** (1960), 131-144.
37. *Sur la théorie générale des demi-anneaux I.* Paris, Séminaire DUBREIL-PISOT, 1960-1961, exposé 24.
38. *Sur la théorie générale des demi-anneaux II.* Paris, Séminaire DUBREIL--PISOT, 1960-1961, exposé 25.
39. *Filtres et réseaux.* Rev. Fac. Ci. Lisboa, **8** (1961), 311-332.
40. *Sur la théorie générale des demi-anneaux.* Publ. Math. Debrecen, **10** (1963), 14-29.
41. *Cours d'Algèbre générale*, vol. I, 494 págs., 1964, Fundação Calouste Gulbenkian, Lisboa.
42. *Sur le demi-anneau des nombres naturels* (em colaboração com a Prof.ª MARIA LUISA GALVÃO). Anais Fac. Ci. Porto, **68** (1965), 5 págs.
43. *Cours d'Algèbre générale*, vol. II, 660 págs., 1967, Fundação Calouste Gulbenkian, Lisboa.
44. *Sur les μ-demi-anneaux.* Math. Zeitschrift, **108** (1968), 10-14.
45. *Sur les idéaux nucléaires d'un demi-anneau.* Rev. Fac. Ci. Lisboa, **11** (1968), 277-293.

46. *Cours d'Algèbre génerale*, vol. I, 2.ª edição, 597 págs, 1969, Fundação Calouste Gulbenkian, Lisboa.
47. *Problèmes de perspectivité dans les ensembles partiellement ordonnés et dans les treillis* (em colaboração com a Prof.ª MARGARITA RAMALHO). Rev. Fac. Ci. Lisboa, **13** (1971), 229-240.
48. *Sur les relations d'invariance dans les treillis* (em colaboração com a Prof.ª MARIA LUISA GALVÃO). Rev. Fac. Ci. Lisboa, **14** (1972), 45-59.
49. *Les μ-demi-anneaux.* Paris, Séminaire DUBREIL-PISOT, 1972, Journées d'Algèbre.
50. *Sur les demi-anneaux réticulés* (em colaboração com a Prof.ª MARIA LUISA GALVÃO). Rev. Fac. Ci. Lisboa, **14** (1973), 195-208.
51. *Cours d'Algèbre générale*, vol. III, 655 págs., 1974, Fundação Calouste Gulbenkian, Lisboa.

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
Lisboa, 1974

FUNCTORS ASSOCIATED WITH VARIETIES OF UNIVERSAL ALGEBRAS (*)

BY

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Fui aluno do Prof. Almeida Costa em 1956-57, na cadeira de Matemáticas Gerais para o 1.º ano do curso de Engenharia Química do Instituto Superior Técnico. No seu curso dava o nosso Professor, a par de matérias consuetudinárias, outras que o eram bastante menos (lembra-me até que causavam algum escândalo na Faculdade de Engenharia): teoria dos grupos e álgebra linear. A verdade é que a beleza intelectual destas teorias, mormente da segunda, me tocou, assim como a alguns colegas. Foi este o meu primeiro contacto com a Álgebra moderna.

Em 1964 voltei a entrar no I. S. T., mas dessa feita na qualidade de assistente — assistente daquela mesma cadeira, que, todavia, já não era regida pelo Prof. Almeida Costa. Passados dois anos meteu-se-me na cabeça procurar saber um pouco mais do que os alunos — empresa menos fácil do que parece à primeira vista. Decidi pedir uma bolsa de estudo e tive de escolher um campo de trabalho. Tal escolha foi feita — confesso-o — um tanto ao acaso, pois que a minha experiência matemática era assaz reduzida. Pude, no entanto, escutar os conselhos e as sugestões de pessoas a quem os pedi e que fizeram o favor de mos dar: sobretudo o saudoso Prof. Sebastião e Silva, o Prof. Dias Agudo — e o Prof. Almeida Costa. Mas creio que não exagero se afirmar que na minha escolha da Álgebra pesou consideravelmente a lembrança das lições que ouvira, dez anos antes, ao Prof. Almeida Costa. Tanto o Prof. Almeida Costa como o Prof. Sebastião e Silva, a quem, por ter com eles maior familiaridade, pedi as duas abonações que me eram requeridas, me auxiliaram com generosidade que jamais poderei esquecer nos trâmites necessários para obtenção da bolsa.

Parti para Inglaterra em Outubro de 67 e por lá me demorei um poucochinho mais de cinco anos. Foi tempo demasiado para, no regresso, ainda poder encontrar vivo o Prof. Sebastião e Silva — o que

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muito me penalizou. O Prof. Almeida Costa, pelo contrário, encontrei-o em excelentes condições físicas e mentais, e seguindo sempre de perto e com o maior interesse as inovações mais recentes dessa ciência de evolução vertiginosa que é a Álgebra. Então mais uma vez se mostrou que a paixão que o Prof. Almeida Costa nutre pela Álgebra se traduz sempre em auxílio e estímulo especiais a quem quer que se dedique a esse campo da Matemática. Foi o Prof. Almeida Costa que me falou na possibilidade de um lugar na Faculdade de Ciências. Foi ele quem fez a proposta ao Conselho. E, uma vez obtida a aprovação do Conselho, foi ele que, como Director que então era da Facutdade, despachou o mais rapidamente possível todos os passos burocráticos que só dele dependiam. Sem o seu empenhamento e despacho ter-se-ia arrastado por largos meses a minha passagem de Assistente a Professor auxiliar.

O homem vive em grupos, mas o Prof. Almeida Costa sabe que nem só de grupos vive o homem. E este seu interesse activo pelos outros não se exerceu apenas, está bem de ver, no meu caso pessoal — que de entre os que lhe passaram pelas mãos seria, talvez, dos que menos circunstâncias teriam a recomendá-los —, senão que tem beneficiado todos os seus colaboradores.

Outros falarão, melhor do que eu saberia, do papel fundamental — creio que se pode e deve dizer: único — que o Prof. Almeida Costa teve na introdução, implantação e difusão da Álgebra moderna em Portugal. Eu acrescentarei apenas ao que já disse que me acostumei a considerar o Prof. Almeida Costa como grande exemplo de amor à ciência e de dedicação e competência profissional. Tais virtudes, conquanto simples de enunciar, são raras em todos os climas e latitudes, e menos frequentes entre nós do que talvez fosse de desejar. É, pois, com muita satisfação que neste momento manifesto publicamente o meu apreço por todas estas qualidades, bem como o reconhecimento e agradecimento mais sinceros por todos os favores recebidos, os quais espero poder pagar numa das moedas em que se liquidam as dívidas das gerações — e que consiste em, na lembrança de tais exemplos, proceder semelhantemente com os vindouros.

INTRODUCTION

Let $\mathcal{C}$ be a variety of groups with multiple operators or Ω-groups (Ω fixed, of course), in the sense of Higgins [5], and let $\mathcal{V}$ be a subvariety of $\mathcal{C}$. We consider all the Ω-groups $B_i \in \mathcal{V}$. Given any $A \in \mathcal{C}$, let α_{ij} denote an Ω-homomorphism $A \to B_i$. We put $V(A) = \bigcap_{i,j}$ ker α_{ij}, where α_{ij} runs through all the homomorphisms from A to all the B_i's. $V(-)$ is a subfunctor of the identity functor $\mathcal{I}$ on $\mathcal{C}$. Following Fröhlich [3] we shall call $V(-)$ the subfunctor associated with $\mathcal{V}$. $V(A)$ is always an ideal of A $(V(A) \lhd A)$. Consequently

we may define $U(A) = A/V(A)$. $U(-)$ is a quotient functor of $\mathcal{I}$. It will be called the quotient functor associated with $\mathcal{V}$. In [3] Fröhlich proved the following

1. Theorem i) For every $A \in \mathcal{C}$, $U(A) \in \mathcal{V}$; also $A \in \mathcal{V}$ if and only if $U(A) = A$, if and only if $V(A) = 0$;

ii) a subfunctor $T(-)$ of $\mathcal{I}$ is associated with some variety $\mathcal{V}$ (i. e., $T = V$ for some variety $\mathcal{V}$) if and only if T preserves surjections and, for every $A \in \mathcal{C}$, $T(A) \lhd A$;

iii) a quotient functor Q of $\mathcal{I}$ is associated with some variety $\mathcal{V}$ if and only if Q is right exact.

Now, Ω-groups are a particular type of Ω-algebras [2, 6]. We have tried to find out whether it was possible to generalize the above theorem so that one would be able to say something about functors and varieties of Ω-algebras. We found that the generalization was not exceedingly difficult, but it wasn't entirely straightforward, either, for a number of reasons. For instance: the concept of ideal cannot be defined for arbitrary Ω-algebras; how can one, then, define the concept of exact sequence which one needs in order to define the concept of exact functor?

We had to replace ideals by equivalence relations and to introduce some new concepts such as the concept of functor which almost preserves surjections, the concept of almost exact sequence, the concept of right almost exact functor.

We considered a category $\mathcal{A}$ of Ω-algebras (Ω fixed, of course) and a variety $\mathcal{V} \subset \mathcal{A}$. Then, for every $A \in \mathcal{A}$, we defined

$$\Phi_A = \{\rho \mid \rho \in \mathcal{C}_\Omega(A),\ A/\rho \in \mathcal{V}\},$$

where $\mathcal{C}_\Omega(A)$ denotes the set of all Ω-congruences or equivalence relations in the algebra A; we put

$$\overline{\mathcal{V}}(\mathcal{V}, A) = \overline{\mathcal{V}}(A) = \bigcap \{\rho \in \Phi_A\},$$

$$\overline{\mathcal{U}}(\mathcal{V}, A) = \overline{\mathcal{U}}(A) = A/\overline{\mathcal{V}}(A).$$

$\overline{\mathcal{V}}(-)$ is a subfunctor of the functor $\mathcal{D} : \mathcal{A} \to \mathcal{A}^2$ defined by $\mathcal{D}(A) = A^2$, $\mathcal{D}(\varphi : A \to B) = \varphi \times \varphi : A \times A \to B \times B$. $\overline{\mathcal{U}}(-)$ is a quotient functor of the identity functor $\mathcal{I}$ in $\mathfrak{A}$. We call $\overline{\mathfrak{V}}(-)$ and $\overline{\mathfrak{U}}(-)$ the functors associated with the variety $\mathfrak{V}$. Then one has the following

2. Theorem: i) $\overline{\mathfrak{U}}(A) \in \mathfrak{V}$ for every $A \in \mathfrak{A}$; also $A \in \mathfrak{V}$ if and only if $\overline{\mathfrak{U}}(A) = A$, if and only if $\overline{\mathfrak{V}}(A) = \Delta_A$;

ii) a subfunctor $\mathfrak{g}(-)$ of $\mathfrak{D}$ is associated with some variety $\mathfrak{V}$ if and only if for every $A \in \mathfrak{A}$, $\mathfrak{g}(A)$ is an Ω-congruence on A and $\mathfrak{g}(-)$ almost preserves surjections;

iii) a quotient functor $\mathcal{I}/\mathfrak{g}(-)$ is associated with some variety $\mathfrak{V}$ if and only if $\mathcal{I}/\mathfrak{g}(-)$ almost preserves Ω-congruences.

1. SETS AND MAPPINGS

Let A and B be sets: $A \times B$ will denote the ordinary Cartesian product; A^2 will stand for $A \times A$ and, in general, $A^{n+1} = A^n \times A$; we shall also put $A^0 = \{\emptyset\}$.

Correspondence: A correspondence from A to B is a subset of $A \times B$.

The set of all correspondences from A to B will be denoted by $\mathrm{Cor}(A, B)$. This is the same as $\mathfrak{B}(A \times B)$, *the Boolean* of $A \times B$, since by definition, for any set $C, \mathfrak{B}(C) = \{X \mid X \subset C\}$.

Given a correspondence $\varphi \in \mathrm{Cor}(A, B)$, say, we denote by φ^{-1} the set

$$\{(y, x) \mid (x, y) \in \varphi\}.$$

Therefore $\varphi^{-1} \in \mathrm{Cor}(B, A)$. In general φ^{-1} is not an inverse.

If $\varphi \in \mathrm{Cor}(A, B)$ and $A' \subset A$ we define

$$A'\varphi = \{y \in B \mid (x, y) \in \varphi \text{ for some } x \in A'\}.$$

Similarly, if $B' \subset B$,

$$B'\varphi^{-1} = \varphi B' = \{x \in A \mid (x, y) \in \varphi \text{ for some } y \in B'\}.$$

If $a \in A$, $b \in B$, we write $a\varphi$, $b\varphi^{-1}$ with an altogether similar meaning.

The elements of $\mathrm{Cor}(A, A)$ will be called *correspondences in* A. For every set A there exists a correspondence

$$\Delta_A = \{(x, x) \mid x \in A\}$$

which is called *the diagonal in* A.

Composition law: One may define a composition law for correspondences in the following way: given two correspondences φ and ψ, one puts

$$\varphi \circ \psi = \{(x, z) \mid (x, y) \in \varphi, (y, z) \in \psi, \text{ for some } y\}.$$

If $\varphi \in \mathrm{Cor}(A, B)$ and $\psi \in \mathrm{Cor}(C, D)$, then $\varphi \circ \psi \in \mathrm{Cor}(A, D)$. It is obvious that $B \cap C = \emptyset$ implies $\varphi \circ \psi = \emptyset$. This composition is associative. Moreover

$$(\varphi \circ \psi)^{-1} = \psi^{-1} \circ \varphi^{-1}; \quad (\varphi^{-1})^{-1} = \varphi.$$

Equivalence relations or congruences: A correspondence φ on a set A will be said to be

i) reflexive if $\varphi \supset \Delta_A$;

ii) symmetric if $\varphi = \varphi^{-1}$;

iii) transitive if $\varphi \circ \varphi \subset \varphi$.

A correspondence which shows these three properties will be said to be an *equivalence relation* or a *congruence.* We shall denote by $\mathfrak{C}(A)$ the set of all congruences on the set A.

Map: By a map φ from a set A to a set B we mean an element of $\mathrm{Cor}(A, B)$ with the property that for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in \varphi$.

The following will be useful. Let $\varphi \in \mathrm{Cor}(A, B)$:

i) φ is a map $\Longleftrightarrow \varphi\varphi^{-1} \supset \Delta_A, \varphi^{-1}\varphi \subset \Delta_B$;
ii) φ is an injective map $\Longleftrightarrow \varphi\varphi^{-1} = \Delta_A, \varphi^{-1}\varphi \subset \Delta_B$;
iii) φ is a surjective map $\Longleftrightarrow \varphi\varphi^{-1} \supset \Delta_A, \varphi^{-1}\varphi = \Delta_B$;
iv) φ is a bijective map $\Longleftrightarrow \varphi\varphi^{-1} = \Delta_A, \varphi^{-1}\varphi = \Delta_B$.

We shall denote by $\mathrm{Map}(A, B)$ the set of all maps from A to B.

Quotients: If $\rho \in \mathcal{C}(A)$, we denote by A/ρ the set of equivalence classes, as usual; A/ρ is called the quotient of A by the congruence ρ; there will then exist a natural map

$$\pi : A \to A/\rho$$

which takes each element of A into the corresponding equivalence class: $a\pi = a\rho$ for every $a \in A$. If a and a' belong to A,

$$a\pi = a'\pi \Leftrightarrow (a, a') \in \rho;$$

π is a surjection.

3. Lemma: i) If φ is a map from A to B, then $\varphi\varphi^{-1}$ is an equivalence relation on A, which we shall call the *equivalence kernel* K_e of $\varphi : K_e\varphi = \varphi\varphi^{-1}$.

ii) If ρ is an arbitrary congruence on A and

$$\pi : A \to A/\rho$$

is the quotient map, then

$$K_e\pi = \pi\pi^{-1} = \rho, \qquad \pi^{-1}\pi = \Delta_{A/\rho}.$$

4. Lemma: *(Universal property of quotient sets):* Let ρ be a congruence on A and let us consider the map $\pi : A \to A/\rho$. If $\varphi \in \mathrm{Map}(A, B)$ and $q = \varphi\varphi^{-1} \supset \rho$, then there exists a unique map $\varphi^* \in \mathrm{Map}(A/\rho, B)$ such that $\pi\varphi^* = \varphi$, namely, $\varphi^* = \pi^{-1}\varphi$; φ^* is injective if and only if $\varphi\varphi^{-1} = \rho$.

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & A/\rho \arrow[d, "\varphi^*"] \\
 & B
\end{tikzcd}$$

5. Lemma: Let $\rho \in \mathcal{C}(A)$ and π be as previously. There exists a $1-1$ correspondence between congruences q on A such that $q \supset \rho$ and congruences q/ρ on A/ρ (two ρ classes are congruent modulo q/ρ if and only if they are contained in the same q class).

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[ddr, "\pi_q"'] & A/\rho \arrow[dd, "\psi"] \arrow[dr, "\pi_{q/\rho}"] & \\
 & & (A/\rho)/(q/\rho) \arrow[dl, "\psi^*"] \\
 & A/q &
\end{tikzcd}$$

In fact, since $q \supset \rho$, there exists a unique ψ that makes the left half of the above diagram commutative. But ψ is a surjection and

$$K_e\psi = \psi\psi^{-1} = \pi^{-1}\pi_q\pi_q^{-1}\pi = \pi^{-1}q\pi = q(\pi \times \pi) = q/\rho,$$

therefore

$$(A/\rho)/(q/\rho) \cong A/q.$$

2. Ω-ALGEBRAS

N-ary operations: For an $n \geq 0$, an n-ary operation on A, or operation of weight n, is a map from A^n to A. A *nullary* operation is a map $A^0 = \{\emptyset\} \to A$: it merely picks an element from A.

Ω-algebra: An Ω-algebra is a pair (A, Ω) where $\Omega = \dot{\bigcup}_{n=0}^{\infty} \Omega_n$ ($\dot{\bigcup}$: disjoint union), Ω_n being a set of n-ary operations on A.

When we want to consider more than one Ω-algebra of the same type at a time, we may take a fixed set of operators $\Omega = \dot{\bigcup}^{\infty} \Omega_n$ and define an Ω-algebra to be a set A together with a collection of operations on A, one operation ω_A for each $\omega \in \Omega$, ω_A having weight n if ω has weight n; ω_A denotes the action of ω in A, but in general one writes ω for ω_A, except if there may be any ambiguity; if $\omega \in \Omega_n$, then one puts

$$a_1 a_2 \cdots a_n \omega = (a_1, a_2, \cdots, a_n)\,\omega_A .$$

From now on, whenever we consider more than one Ω-algebra at a time, we shall suppose that they are of the same type.

Ω-homomorphism: If A and B are Ω-algebras, an Ω-homomorphism from A to B is a map $\varphi \in \mathrm{Map}(A, B)$ satisfying, for every $n \geqq 0$, every $\omega \in \Omega_n$ and any $a_1, a_2, \cdots, a_n \in A$,

$$a_1 a_2 \cdots a_n \omega \varphi = a_2 \varphi \cdots a_n \varphi \omega ,$$

or equivalently: $(a_1, a_1 \varphi) \in \varphi, \cdots, (a_n, a_n \varphi) \in \varphi \Rightarrow (a_1 \cdots a_n \omega, a_1 \varphi \cdots a_n \varphi \omega) \in \varphi$.

If $\omega \in \Omega_0$, we shall call ω_A a *constant* of A. In this case, any Ω-homomorphism from A to B will send ω_A into ω_B.

Ω-subalgebra: If A is an Ω-algebra, $B \subset A$ is an Ω-subalgebra of A if, for every $n \geqq 0$, every $\omega \in \Omega_n$, and every $a_1, \cdots, a_n \in B$,

$$a_1 \cdots a_n \omega \in B .$$

In particular B must contain all the the constants of A. The empty set is an Ω-subalgebra of A if and only if $\Omega_0 = \phi$. Also, a subset B of A is an Ω-subalgebra of A if and only if $\Delta_B \in \mathrm{Cor}_\Omega(A, A)$ where $\mathrm{Cor}_\Omega(A, A)$ denotes the set of all Ω-correspondences on A, which are defined in the obvious way.

$\mathcal{B}_\Omega(A)$ will denote the set of all Ω-subalgebras of A.

Ω-congruence: An Ω-congruence on the Ω-algebra A is a congruence on A that is an Ω-subalgebra of A^2, i. e., $\rho \in \mathcal{C}(A)$ is an Ω-congruence on A if, for every $n \geqq 0$, for every $\omega \in \Omega_n$ and every $a_1, \ldots, a_n, a'_1, \ldots, a'_n \in A$,

$$(a_1, a'_1) \in \rho, \ldots, (a_n, a'_n) \in \rho \Rightarrow (a_1 \ldots a_n \omega, a'_1 \ldots a'_n \omega) \in \rho$$

We denote the set of Ω-congruences on A by $\mathcal{C}_\Omega(A)$.

Algebraic closure lattice: Let T be any set and let us take $\mathcal{L} \subset \mathcal{B}(T)$ such that

i) $T \in \mathcal{L}$,

ii) if $A_\lambda \in \mathcal{L}$ for $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} A_\lambda \in \mathcal{L}$.

$\mathcal{L}$ is an ordered set with respect to inclusion and it actually is a complete lattice. In fact, if $A_\lambda \in \mathcal{L}$ for $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is the infimum of the subset $\{A_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{L}$ with respect to the inclusion. On the other hand, there exist sets $S \in \mathcal{L}$ which contain all the $A'_\lambda s$; the intersection of all such sets is a member of $\mathcal{L}$ which contains all the $A'_\lambda s$, and it clearly is the supremum of $\{A_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{L}$. We call such a lattice a *closure lattice* in the set T, or an *intersection lattice* in T.

In some cases one has the additional property: if $A_\lambda \in \mathcal{L}$ for $\lambda \in \Lambda$ and if the A_λ form a non-empty chain with respect to inclusion, then $\bigcup_{\lambda \in V} A_\lambda \in \mathcal{L}$. Closure lattices with this additional property are called *algebraic closure lattices.*

It is important the fact that, for any Ω-algebra A, $\mathcal{B}_\Omega(A)$ is an algebraic closure lattice in A. It may be proved that so is $\mathcal{C}(A)$ and consequently $\mathcal{C}_\Omega(A)$.

Quotient Ω-algebra: If A is an Ω-algebra and $\rho \in \mathcal{C}_\Omega(A)$, then A/ρ is an Ω-algebra, a quotient algebra, in a natural way; given $a \in A$, we write $\bar{a} = a\rho$; let $a_1, \cdots, a_n \in A$, $\omega \in \Omega_n$; then we put

$$\bar{a}_1 \cdots \bar{a}_n \omega = \overline{a_1 \cdots a_n \omega} .$$

Direct product Ω-algebras: With any family $\{A_\lambda\}_{\lambda \in \Lambda}$ of Ω-algebras a direct product is associated which we define as follows: let Π be the cartesian product of the A_λ regarded as sets, with projections $\pi_\lambda : \Pi \to A_\lambda$; then any element $a \in \Pi$ is completely determined by its components $a\pi_\lambda$, and conversely any choice of elements $a(\lambda) \in A_\lambda$ defines a unique element a of Π by the equalities $a\pi_\lambda = a(\lambda)$ for every $\lambda \in \Lambda$; therefore, if $a_1, \cdots, a_n \in \Pi$ and $\omega \in \Omega_n$, we can define $a_1 \cdots a_n$ by the equalities

$$a_1 \cdots a_n \omega \pi_\lambda = a_1 \pi_\lambda \cdots a_n \pi_\lambda \omega ;$$

in this way an Ω-algebra structure is defined on Π, and it is clear that the projections are Ω-homomorphisms.

For the definitions and results presented so far we refer to |2| and |6|.

3. THE FUNCTORS $\mathfrak{q}(\mathcal{C},—)$ AND $\mathcal{R}(\mathcal{C},—)$

Let $\mathcal{A}$ be a category of Ω-algebras and let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$ which is closed with respect to isomorphic copies. Given an Ω-algebra $A \in \mathcal{A}$, we consider

$$\Phi_A = \{\rho \mid \rho \in \mathcal{C}_\Omega(A), A/\rho \in \mathcal{C}\}.$$

Let us put

$$\mathfrak{q}(\mathcal{C}, A) = \mathfrak{q}(A) = \mathfrak{q}_A = \bigcap \{\rho \in \Phi_A\}$$

(we shall omit the $\mathcal{C}$ in the notation when there is no ambiguity).

Then $\mathfrak{q}_A$ is an Ω-congruence on A. If $\Phi_A = \phi$, we put $\mathfrak{q}_A = A^2$, which accords with a usual convention. In this way it is possible to define, for any $A \in \mathcal{A}$,

$$\mathcal{R}(\mathcal{C}, A) = \mathcal{R}(A) = A/\mathfrak{q}_A.$$

Residually $\mathcal{C}$-algebra: We shall say that A is a residually $\mathcal{C}$-algebra if $\mathfrak{q}_A = \Delta_A$, i.e., if $\mathcal{R}(A) = A$.

We remark that

$$A \in \mathcal{C} \Rightarrow A/\Delta_A \in \mathcal{C} \Rightarrow \Delta_A \in \Phi \Rightarrow \mathfrak{q}_A = \Delta_A,$$

therefore $\mathcal{C} \subset \mathcal{R}\mathcal{C}$, *if we denote by $\mathcal{R}\mathcal{C}$ the class of all residually $\mathcal{C}$-algebras and their isomorphic copies.*

If $\mathcal{C}$ is any category of Ω-algebras we shall denote by

$S\mathcal{C}$ the category of all Ω-algebras isomorphic to subalgebras of algebras in $\mathcal{C}$;

$Q\mathcal{C}$ the category of all Ω-algebras isomorphic to quotients of algebras in $\mathcal{C}$;

$P\mathcal{C}$ the category of all Ω-algebras isomorphic to products of algebras in $\mathcal{C}$.

Closure operations: S, Q and P are closure operations in the following sense. Let A be a set and F a mapping of $\mathcal{P}(A)$ into itself. F is said to be a closure operator if it enjoys the following properties for any $X, Y \in \mathcal{P}(A)$:

i) if $X \subset Y$, then $F(X) \subset F(Y)$,

ii) $X \subset F(X)$,

iii) $FF(X) = F(X)$.

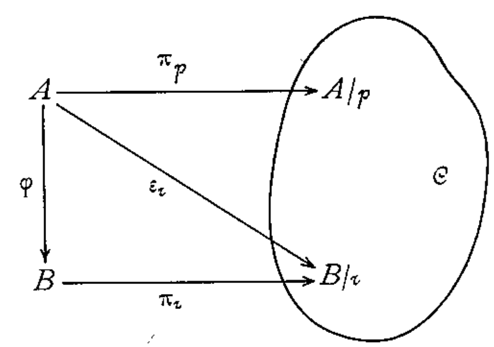
We return to our subcategory $\mathcal{C} \subset \mathcal{A}$ which is closed with respect to isomorphic copies.

6. Proposition: If $\mathcal{C}$ is S-closed, then $\mathfrak{q}(—)$ is a functor $\mathfrak{q}(—): \mathcal{A} \to \mathcal{A}^2$, which actually is a subfunctor of the functor $\mathcal{D}$ defined by $\mathcal{D}(A) = A^2$, $\mathcal{D}(\varphi: A \to B) = \varphi \times \varphi: A \times A \to B \times B$.

Proof: We have seen that $\mathfrak{q}_A$ is an Ω-congruence on A for every $A \in \mathcal{A}$, therefore $\mathfrak{q}_A$ is an Ω-subalgebra of A^2 for every $A \in \mathcal{A}$.

We consider an Ω-homomorphism $\varphi: A \to B$: this homomorphism determines the Ω-homomorphism $\varphi \times \varphi: A \times A \to B \times B$. We claim that $(\varphi \times \varphi) \mid \mathfrak{q}_A: \mathfrak{q}_A \to \mathfrak{q}_B$. To see this we consider $\Phi_A = \{\rho \mid \rho \in \mathcal{C}_\Omega(A), A/\rho \in \mathcal{C}\}$ and Φ_B similarly defined. By definition

$$\mathfrak{q}_A = \bigcap \{\rho \in \Phi_A\}, \qquad \mathfrak{q}_B = \bigcap \{\tau \in \Phi_B\}.$$



For every $\tau \in \Phi_B$ the corresponding Ω-homomorphism $\pi_\tau: B \to B/\tau$ determines an Ω-homomorphism $\varepsilon_\tau: A \to B/\tau$ just by composing it with $\varphi: \varepsilon_\tau = \varphi\pi_\tau$. We shall show that $\varepsilon_\tau\varepsilon_\tau^{-1} \in \Phi_A$. In fact $A/\varepsilon_\tau\varepsilon_\tau^{-1}$ is isomorphic to a subalgebra of B/σ; since $\mathcal{C}$ is, by hypothesis, S-closed and closed with respect to isomorphic copies, $A/\varepsilon_\sigma\varepsilon_\sigma^{-1} \in \mathcal{C}$. Consequently, each $\sigma \in \Phi_B$ determines a $\rho_\sigma \in \Phi_A$. We shall write ρ' for the ρ's of the form $\varepsilon_\sigma\varepsilon_\sigma^{-1}$ for some $\sigma \in \Phi_B$. Then

$$(x, y) \in \bigcap \{\rho \in \Phi_A\} = \eta_A \Rightarrow (x, y) \in \rho \text{ for every } \rho \in \Phi_A \Rightarrow$$

$$(x, y) \in \rho' \text{ for every } \rho' \in \Phi_A \Rightarrow (x\varphi, y\varphi) \in \sigma \text{ for every } \sigma \in \Phi_B \Rightarrow$$

$$(x\varphi, y\varphi) \in \bigcap \{\sigma \in \Phi_B\} = \eta_B .$$

It is clear that $\eta(-)$ satisfies the remaining specifications in order to be a functor.

7. Proposition: If $\eta(-)$ is a functor, then $\mathcal{R}(-)$ is a functor $\mathcal{R}(-) : \mathcal{A} \to \mathcal{A}$.

Proof: If $\varphi : A \to B$ is an Ω-homomorphism, then $\varphi\times\varphi : \eta_A \to \eta_B$. Then φ induces $\varphi^* : \mathcal{R}(A) = A/\eta_A \to B/\eta_B = \mathcal{R}(B)$,

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & A/\eta_A \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi^*} \\ B & \xrightarrow{\pi_B} & B/\eta_B \end{array} ,$$

for

$$Ke(\varphi\pi_B) = (\varphi\pi_B)(\varphi\pi_B)^{-1} = \varphi\pi_B\pi_B^{-1}\varphi^{-1} =$$

$$= \varphi\eta_B\varphi^{-1} \supset \varphi\eta_A(\varphi\times\varphi)\varphi^{-1} \supset \eta_A\varphi\varphi^{-1} \supset \eta_A = Ke\,\pi_A .$$

8. Proposition: Let $\mathcal{C}$ be a full subcategory of the category $\mathcal{A}$ as before. We shall suppose that $\mathcal{C}$ is isomorphically closed, but it may or may not be S-closed. Then $\mathcal{R}(A) \in \mathcal{R}\mathcal{C}$ for every $A \in \mathcal{A}$.

Proof: We want to show that, for every $A \in \mathcal{A}$, $\eta(A/\eta_A) = \Delta_{A/\eta_A}$. We shall drop the subscript A of η for the time being. By definition $\rho \supset \eta$, for every $\rho \in \Phi_A$. We shall distinguish the

different elements of Φ_A by means of an index i taken from an index set I. As we know, ρ_i/η is an Ω-congruence on A/η, and $(A/\eta)/(\rho_i/\eta) \cong A/\rho_i \in \mathcal{C}$. But this means that $\rho_i/\eta \in \Phi_{A/\eta}$. Now

$$\bigcap_{i\in I}\left(\frac{\rho_i}{\eta}\right) = \frac{\bigcap_{i\in I}\rho_i}{\eta} = \frac{\eta}{\eta} = \Delta_{A/\eta} .$$

We only have to show the first equality. We remark that $\rho/\eta = \rho(\pi_\eta\times\pi_\eta)$. If $(a, a') \in \bigcap_{i\in I}\rho_i(\pi_\eta\times\pi_\eta)$, then for every $i \in I$ there exists $(a_i, a'_i) \in \rho_i$ such that $(a_i, a'_i)(\pi_\eta\times\pi_\eta) = (a, a')$.

If we consider two different subscripts i and j, then

$$a_i\pi_\eta = a_j\pi_\eta = a , \qquad a'_i\pi_\eta = a'_j\pi_\eta = a',$$

therefore, for every i and j, $(a_i, a_j) \in \eta$, $(a'_i, a'_j) \in \eta$. Therefore, $(a_j, a_i) \in \rho_i$, $(a'_i, a'_j) \in \rho_i$, and since $(a_i, a'_i) \in \rho_i$ one has $(a_j, a'_j) \in \rho$. Consequently, for every j, $(a_j, a'_j) \in \bigcap_{i\in I}\rho_i$, and so

$$(a, a') \in (\bigcap_{i\in I}\rho_i)(\pi_\eta\times\pi_\eta) = (\bigcap_{i\in I}\rho_i)/\eta .$$

Corollary: If $\mathcal{R}\mathcal{C} = \mathcal{C}$, then the following statements are equivalent: i) $A \in \mathcal{C}$; ii) $\eta_A = \Delta_A$; iii) $\mathcal{R}(A) = A$.

In particular, if $\mathcal{C}$ is SP-closed, then $A \in \mathcal{C}$ if and only if $\eta_A = \Delta_A$. In fact, it may be proved that if $\mathcal{C}$ is SP-closed, then it is $\mathcal{R}$-closed (cf. [2] and [6]).

Let $\vartheta(-)$ be a subfunctor of $\mathcal{D}$ such that $\vartheta(A)$ is an Ω-congruence in A for every $A \in \mathcal{A}$. If $\varphi : A \to B$ is an Ω-homomorphism, then $\varphi\times\varphi : \vartheta_A \to \vartheta_B$. In general $\vartheta_A(\varphi\times\varphi) \neq \vartheta_B$ even if φ is surjective for, in this favourable case, $\vartheta_A(\varphi\times\varphi)$ is reflexive and symmetric, but not transitive in general. Let us suppose that φ is in fact a surjection and let us denote by $\Gamma_{\mathcal{C}_\Omega(B)}(\vartheta_A(\varphi\times\varphi))$ the Ω-congruence in B generated by $\vartheta_A(\varphi\times\varphi)$, i. e., the minimal Ω-congruence in B which contains $\vartheta_A(\varphi\times\varphi)$. Since $\vartheta_A(\varphi\times\varphi)$ is, in these circumstances, reflexive and symmetric, in order to obtain transitivy it will be enough to consider the different powers of the Ω-correspondence $\vartheta_A(\varphi\times\varphi)$ in B and take their union:

$$\Gamma_{\mathcal{C}_\Omega(B)}(\vartheta_A(\varphi\times\varphi)) = \bigcup_{n=0}^{\infty}(\vartheta_A(\varphi\times\varphi))^n = \overline{\widetilde{\vartheta_A(\varphi\times\varphi)}} .$$

The law of composition for the powers of $q_A(\varphi\times\varphi)$ is the one for correspondences, of course, and we make the convention that $(q_A(\varphi\times\varphi))^0=\Delta_B$.

Now, φ being a surjection, it may happen that $\overline{\widetilde{q_A(\varphi\times\varphi)}}=q_B$. If this does happen we shall say that $q(-)$ *almost preserved the surjection*. If this happens whenever φ is a surjection in $\mathcal{C}$, then $q(-)$ will be said to *almost preserve surjections*.

We consider the surjective homomorphism $\varphi: A\to B$ along with q_A, q_B, $q_A(\varphi\times\varphi)$, $\overline{\widetilde{q_A(\varphi\times\varphi)}}$; φ induces

$$\begin{tikzcd} A \arrow[r, "\pi_A"] \arrow[d, "\varphi"'] & A/q_A \arrow[d, "\varphi^*"] \\ B \arrow[r, "\pi_B"] & B/\overline{\widetilde{q_A(\varphi\times\varphi)}} \end{tikzcd}$$

the homomorphism φ^* that makes the present diagram commutative. If $\mathcal{C}$ is $\mathcal{R}$-closed, $A/q_A\in\mathcal{C}$; if $\mathcal{C}$ is Q-closed, $B/\overline{\widetilde{q_A(\varphi\times\varphi)}}\in\mathcal{C}$ (φ^* is surjective). Therefore, $\overline{\widetilde{q_A(\varphi\times\varphi)}}\in\Phi_B$ and so $q_B\subset\overline{\widetilde{q_A(\varphi\times\varphi)}}$. But $q_A(\varphi\times\varphi)\subset q_B$, therefore $\overline{\widetilde{q_A(\varphi\times\varphi)}}\subset q_B$. We conclude that $\overline{\widetilde{q_A(\varphi\times\varphi)}}=q_B$. In view of what has been said we may state

9. Proposition: If $\mathcal{C}$ is S, R, and Q closed, then $q(-)$ almost preserves surjections.

We want, now, to consider Ω-words and Ω-word algebras in order to define varieties. According to P. M. Cohn |2| this may be done as follows.

Ω-word and Ω-word algebra: Given an operator domain Ω and a set X (which we shall suppose, without loss of generality, to be disjoint from Ω), the *Ω-word algebra on X* is defined thus: by an *Ω-row in X* we shall mean a finite sequence (i. e., an n-tuple for $n\geq 1$) of elements of $\Omega\cup X$; on the set $W(\Omega, X)$ of all Ω-rows in

X (which contains both Ω and X) we define an Ω-algebra structure by juxtaposition: if $\omega\in\Omega_n$ and $a_i\in W(\Omega, X)$ $(i=1,\cdots,n)$, with

$$a_i=(a_{i1},\cdots,a_{ik_i})\qquad (a_{ij}\in\Omega\cup X),$$

then

$$a_1\cdots a_n\,\omega=a_{11}\cdots a_{1k_1}a_{21}\cdots a_{nk_n}\,\omega=$$
$$=(a_{11},\cdots,a_{1k_1}a_{21},\cdots,a_{nk_n},\omega).$$

$W(\Omega, X)$ with this structure is then the *algebra of Ω-rows* in X. The subalgebra of $W(\Omega, X)$ generated by X is called the *Ω-word algebra on X* which we denote by $W=W_\Omega(X)$. Its elements are called *Ω-words on X* and X is called its *alphabet*. $W_\Omega(X)$ is essentially determined by the cardinality of X.

Variety: If A is any Ω-algebra and $\alpha: W\to A$ a homomorphism, given any $w\in W$ we call $w\alpha$ a value of w in A. A *law* or *identity* over Ω in the alphabet X is a pair $(w_1, w_2)\in W^2$, or sometimes the equation

$$w_1=w_2 \tag{1}$$

formed from that pair. One says that the law (1) holds in A, or that A satisfies (1), if for every homomorphism $\alpha: W\to A$ one has $w_1\alpha=w_2\alpha$. If Σ is a set of laws, then the variety $\mathcal{V}(\Sigma)$ defined by Σ is the class of all Ω-algebras which satisfy all the laws in Σ. A variety of Ω-algebras is the class of all Ω-algebras satisfying some given set of laws.

Varieties as defined before depend on the alphabet X. However, it may be proved that all varieties may be obtained by using any fixed alphabet which is infinite, but otherwise arbitrary. Usually one takes as standard alphabet a countable set $X_0=\{x_1, x_2, \cdots\}$ which is indexed by the positive integers.

It may also be shown |2| that varieties are S, P and Q closed and since, as we have already remarked, SP-closed implies R-closed, one has the following corollary of the last Proposition:

Corollary: If $\mathcal{C}$ is a variety, then $q(-)$ almost preserves surjections.

10. Proposition: If $\mathfrak{g}(-)$ is a subfunctor of the functor $\mathfrak{D}$ that almost preserves surjections and such that, for every $A \in \mathcal{A}$, $\mathfrak{g}(A)$ is an Ω-congruence on A, then

i) $\mathfrak{g}(-)$ defines the variety $\mathcal{V} = \{A \in \mathcal{A} \mid \mathfrak{g}(A) = \Delta_A\}$;

ii) $\mathfrak{g}(A) = \mathfrak{q}(\mathcal{V}, A) = \cap\{\mathfrak{p} \mid \mathfrak{p} \in \Phi(\mathcal{V}, A)\}$ for every $A \in \mathcal{A}$.

Proof: First, we want to show that $\mathcal{V}$ is S, P and Q closed for then it will be a variety [2]. But before we do this, we note that in a category $\mathcal{A}$ with direct products the two following conditions are equivalent:

a) if $\{A_\lambda\}_{\lambda \in \Lambda}$ is an indexed set of Ω-algebras such that $A_\lambda \in \mathcal{C}$, for every $\lambda \in \Lambda$, then $\prod\limits_{\Lambda} A_\lambda \in \mathcal{C}$;

b) if $\{A_\lambda, f_\lambda\}_{\lambda \in \Lambda}$ is an indexed set of pairs of Ω-algebras $A_\lambda \in \mathcal{C}$ and Ω-surjections $f_\lambda : A \to A_\lambda$, with A fixed in $\mathcal{A}$, and if $\cap\{\mathfrak{p}_\lambda \mid \lambda \in \Lambda\} = \Delta_A$, where $\mathfrak{p}_\lambda = f_\lambda f_\lambda^{-1}$, then $A \in \mathcal{C}$.

We shall prove that $\mathcal{V}$ is closed with respect to condition *b*). In fact, let us consider the circumstances described in *b*); then, for every $\lambda \in \Lambda$,

$$f_\lambda \times f_\lambda : \mathfrak{g}_A \to \Delta_{A_\lambda},$$

i. e., if $(a, b) \in \mathfrak{g}_A$, then $a f_\lambda = b f_\lambda$, i. e., $(a, b) \in \mathfrak{p}_\lambda$ for every $\lambda \in \Lambda$. Therefore

$$(a, b) \in \mathfrak{g}_A \Rightarrow (a, b) \in \bigcap_{\lambda \in \Lambda} \mathfrak{p}_\lambda = \Delta_A \Rightarrow \mathfrak{g}_A \subset \Delta_A \Rightarrow$$
$$\mathfrak{g}_A = \Delta_A \Rightarrow A \in \mathcal{V}.$$

Let $\varphi : A \to B$ be an injection with $B \in \mathcal{V}$; then $\mathfrak{g}_B = \Delta_B$ and

$$\varphi \times \varphi : \mathfrak{g}_A \to \mathfrak{g}_B = \Delta_B,$$

therefore $\mathfrak{g}_A \subset Ke\,\varphi = \varphi\varphi^{-1} = \Delta_A$, i. e., $\mathfrak{g}_A = \Delta_A$ and so $A \in \mathcal{V}$: This shows that $\mathcal{V}$ is S-closed.

Let $\varphi : A \to B$ be a surjection with $A \in \mathcal{V}$; then $\mathfrak{g}_A = \Delta_A$ and $\varphi \times \varphi : \Delta_A \to \mathfrak{g}_B$; by hypothesis $\overline{\Delta_A \widetilde{(\varphi \times \varphi)}} = \mathfrak{g}_B$; but $\Delta_A(\varphi \times \varphi) \subset \Delta_B$

and so $\overline{\Delta_A \widetilde{(\varphi \times \varphi)}} \subset \Delta_B$; therefore $\mathfrak{g}_B = \Delta_B$, i. e., $B \in \mathcal{V}$, and so $\mathcal{V}$ is Q-closed.

Now we turn our attention to point ii) of Proposition 10. By hypothesis $\mathfrak{g}_A$ is an Ω-congruence on A, and so we may consider the natural surjection $\pi_{\mathfrak{g}} : A \to A/\mathfrak{g}_A$. One has the map

$$\pi_{\mathfrak{g}} \times \pi_{\mathfrak{g}} : \mathfrak{g}_A = \mathfrak{g}(A) \to \mathfrak{g}(A/\mathfrak{g}_A)$$

and so

$$\mathfrak{g}(A)(\pi_{\mathfrak{g}} \times \pi_{\mathfrak{g}}) \subset \mathfrak{g}(A/\mathfrak{g}_A);$$

but

$$\mathfrak{g}(A)(\pi_{\mathfrak{g}} \times \pi_{\mathfrak{g}}) \subset \Delta_{A/\mathfrak{g}_A},$$

and therefore

$$\overline{\mathfrak{g}(A)\widetilde{(\pi_{\mathfrak{g}} \times \pi_{\mathfrak{g}})}} = \Delta_{A/\mathfrak{g}_A}.$$

By hypothesis, $\overline{\mathfrak{g}(A)\widetilde{(\pi_{\mathfrak{g}} \times \pi_{\mathfrak{g}})}} = \mathfrak{g}(A/\mathfrak{g}_A)$, consequently $\mathfrak{g}(A/\mathfrak{g}_A) = \Delta_{A/\mathfrak{g}_A}$, i. e., $A/\mathfrak{g}_A \in \mathcal{V}$; this means that $\mathfrak{g}_A \in \Phi_A$. On the other hand, given any $\mathfrak{p} \in \Phi_A$, there exists a natural surjection

$$\pi_{\mathfrak{p}} : A \to A/\mathfrak{p},$$

whith $A/\mathfrak{p} \in \mathcal{V}$; this tells us that $\mathfrak{g}(A)(\pi_{\mathfrak{p}} \times \pi_{\mathfrak{p}}) \subset \mathfrak{g}(A/\mathfrak{p}) = \Delta_{A/\mathfrak{p}}$, therefore $\mathfrak{g}(A) \subset \mathfrak{p}$, and consequently $\mathfrak{g}(A) = \mathfrak{q}(\mathcal{V}, A)$.

Now we see that, given a variety $\mathcal{V}$, we can associate with it the functor $\mathfrak{q}(\mathcal{V}, -)$ and call it *the subfunctor of* $\mathfrak{D}$ *associated whith* $\mathcal{V}$. For we have just proved that

11. Proposition: A subfunctor $\mathfrak{g}(-)$ of $\mathfrak{D}$ will be associated with a variety if and only if, for every $A \in \mathcal{A}$, $\mathfrak{g}(A)$ is an Ω-congruence on A and $\mathfrak{g}(-)$ almost preserves surjections.

4. ALMOST EXACT SEQUENCES OF Ω-ALGEBRAS

4. 1. Generation of congruences

Let A be an Ω-algebra and let R be a subset of A^2. We want to determine the smallest congruence on A which contains R, i. e., the closure $\Gamma_{\mathcal{C}(A)}(R)$ of R in $\mathcal{C}(A)$.

12. PROPOSITION: i) $\Gamma_{\mathcal{C}(A)}(R) = \widetilde{R \cup R^{-1}} = \bigcup_{n=0}^{\infty} (R \cup R^{-1})^n$, i.e., $a \equiv b \ (\mathrm{mod}\, \Gamma_{\mathcal{C}(A)}(R))$ if and only if, for some $n > 0$, there exist $a_0, a_1, \cdots, a_n \in A$ such that $a = a_0$, $b = a_n$ and $(a_i, a_{i+1}) \in R \cup R^{-1}$, $0 \leq i \leq n-1$ (we have made the convention: $(R \cup R^{-1})^0 = = \Delta_A$); ii) if $S = R \cup \Delta_A$, then $\Gamma_{\mathcal{C}(A)}(R) = \widetilde{S S^{-1}} = \bigcup_{n=1}^{\infty} (S S^{-1})^n$.

13. PROPOSITION: Let $R \subset A^2$ and let $\overline{R} = \Gamma_{\mathcal{B}_\Omega(A^2)}(R)$ be the Ω-subalgebra of A^2 generated by R. If $R \supset \Delta_A$, then $\Gamma_{\mathcal{C}_\Omega(A)}(R) = \widetilde{\overline{R}}$.

PROOF. By hypothesis, $\overline{R}$ is an Ω-subalgebra of A^2 containing Δ_A. Therefore $\Gamma_{\mathcal{C}_\Omega(A)}(R) = \bigcup_{n=1}^{\infty} (\overline{R}\, \overline{R}^{-1})^n$; $P = \overline{R}\, \overline{R}^{-1}$ is an Ω-subalgebra of A^2, since inverses and products of Ω-subalgebras of A^2 are Ω-algebras; for the same reason, P^n is an Ω-subalgebra of A^2, for every $n > 1$; moreover,

$$P \subset P^2 \subset P^3 \subset \cdots \subset P^n \subset \cdots,$$

since $P \supset \Delta_A$, i.e., we have a chain. Since the lattice of Ω-subalgebras is an algebraic closure lattice, $\widetilde{\overline{R}\, \overline{R}^{-1}} := \bigcup_{n=1}^{\infty} P^n$ is an Ω-subalgebra. But it is an equivalence relation, as well; it is, therefore, an Ω-congruence, and the smallest one containing R.

For the last two propositions we refer to |6|.

4.2. Short almost exact sequences of Ω-algebras (saes)

Given a sequence

$$\cdots \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow \cdots \tag{1}$$

of ordinary groups and group homomorphisms, one says that the sequence is exact at B if $\mathrm{im}\,\alpha = \ker \beta$, i.e., one requires α to be *normal* and also the normal subgroup $\mathrm{im}\,\alpha$ of B to be precisely the kernel of β. The same definition applies to any type of Ω-group: one just has to replace the concept of normal subgroup by that of *ideal*.

However, when dealing with Ω-algebras, one does not have the concept of normal subalgebra, nor is there any easy analogue, as happens in the case of Ω-groups. We must therefore look for more general concepts. In the case of ordinary groups (and in the case of Ω-groups, if we substitute the concept of ideal for that of normal subgroup), there is a $1-1$ correspondence between the set of normal subgroups of a given group and the set of congruences, or equivalence relations, on the same group. Consequently, the natural thing to do, if one wishes to try to generalise the concept of exactness, is, given a sequence of Ω-algebras and Ω-homomorphisms of type (1), to consider, on one hand, the Ω-congruence on B^2 generated by the image of A^2 under $\alpha \times \alpha$, on the other hand the equivalence kernel of β, and then require them to be equal. It is clear that in doing so we loosen the conditions imposed on α and β, to some extent.

Almost exactness: Given a sequence of Ω-algebras and Ω-homomorphisms of type (1), if we call ρ_α the Ω-congruence generated on B^2 by $(A\alpha, A\alpha)$, we shall say that the sequence is almost exact at B if

$$\rho_\alpha = \overline{\widetilde{(A\alpha, A\alpha)}} = \beta\beta^{-1}.$$

14. PROPOSITION: If the sequence (1) is almost exact at B, then C has a one-element subalgebra.

PROOF. For every $a_1, a_2 \in A$ we have

$$(a_1\alpha, a_2\alpha) \in (A\alpha, A\alpha) \subset \rho_\alpha = \beta\beta^{-1},$$

therefore $a_1\alpha\beta = a_2\alpha\beta = c \in C$; since a_1 and a_2 were arbitrary, one has $A\alpha\beta = \{c\}$; but $\alpha\beta$ is an Ω-homomorphism, therefore $\{c\}$ is a one-element subalgebra.

We remark that, if $\Omega_0 \neq \emptyset$, an Ω-algebra cannot have more then one one-element subalgebra.

We shall define a *short almost exact sequence* as a sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{2}$$

of Ω-algebras and Ω-homomorphisms satisfying:

i) $\alpha\alpha^{-1}=\Delta_A$, i. e., α is an injection;

ii) $\beta^{-1}\beta=\Delta_C$, i. e., β is a surjection;

iii) $\rho_\alpha=\beta\beta^{-1}$, i. e., the sequence is almost exact at B.

If we consider (2) as a sequence of ordinary groups and group homomorphisms, the meaning of the three previous conditions is the following:

i) means that α is a monomorphism;

ii) means that β is an epimorphism;

iii) means that the normal subgroup of B generated by $\operatorname{im}\alpha$ is precisely $\ker\beta$;

this shows more clearly why we say that the sequence is *almost* exact when it satisfies i), ii) and iii).

Given a category $\mathfrak{A}$ of Ω-algebras, a functor $F:\mathfrak{A}\to\mathfrak{A}$ will be said to be *right almost exact* if, whenever we are given a saes like (2), satisfying i), ii) and iii), then we also have a sequence

$$AF\xrightarrow{\alpha F}BF\xrightarrow{\beta F}CF$$

satisfying

i′) $(\beta F)^{-1}(\beta F)=\Delta_{CF}$, i. e., BF is a surjection;

ii′) $\rho_{\alpha F}=(\beta F)(\beta F)^{-1}$, i. e., the sequence is almost exact at BF.

5. THE FUNCTORS $\overline{\mathfrak{V}}$ AND $\overline{\mathfrak{U}}$

We have seen that, given a variety $\mathfrak{V}$, it is possible to define the functor associated with $\mathfrak{V}$, and this is $q(\mathfrak{V},-)$; $q(\mathfrak{V},-)$ exhibits some peculiar properties which derive from the fact that $\mathfrak{V}$ is a variety and not just any category of Ω-algebras. From now on we shall be dealing only with functors associated with varieties and we shall denote $q(\mathfrak{V},-)$ by $\overline{\mathfrak{V}}(-)$. We shall also put $\mathfrak{R}(\mathfrak{V},-)=\overline{\mathfrak{U}}$; $\overline{\mathfrak{U}}$ is of the form $\mathfrak{I}/\overline{\mathfrak{V}}$, i. e., a quotient functor of $\mathfrak{I}$(*) ($QF\mathfrak{I}$, in short). Therefore, given $\mathfrak{V}$, $\overline{\mathfrak{U}}$ is the $QF\mathfrak{I}$ associated with $\mathfrak{V}$.

We are going to examine the exactness of $\overline{\mathfrak{U}}$, but in order to do that we need the results of

15. Lemma: If

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\pi_A} & & \downarrow{\scriptstyle\pi_B} \\ \hat{A} & \xrightarrow{\hat{\alpha}} & \hat{B} \end{array},$$

where $\hat{A}$ and $\hat{B}$ are quotients of A and B respectively, is a commutative diagram of Ω-homomorphisms, then

i) $\rho_\alpha(\pi_B\times\pi_B)\subset\rho_{\hat{\alpha}}$;

ii) $\overline{\rho_\alpha\widetilde{(\pi_B\times\pi_B)}}=\rho_{\hat{\alpha}}$;

iii) for every $(u,u')\in\overline{(\hat{A}\hat{\alpha},\hat{A}\hat{\alpha})\cup\Delta_{\hat{B}}}$ there exists $(b,b')\in\overline{(A\alpha,A\alpha)\cup\Delta_B}$ such that $(b,b')(\pi_B\times\pi_B)=(u,u')$.

Proof: i) $(b,b')\in\rho_\alpha$ implies that, for some $n\geq 1$, there exist $b_0,b_1,\cdots,b_n\in B$ such that $b_0=b$, $b_n=b'$ and $(b_i,b_{i+1})\in\in\overline{(A\alpha,A\alpha)\cup\Delta_B}$, $0\leq i\leq n-1$. But

$$\overline{(A\alpha,A\alpha)\cup\Delta_B}(\pi_B\times\pi_B)=\overline{((A\alpha,A\alpha)\cup\Delta_B)(\pi_B\times\pi_B)}=$$
$$=\overline{(\hat{A}\hat{\alpha},\hat{A}\hat{\alpha})\cup\Delta_{\hat{B}}},$$

since the bar indicates composition by means of elements of Ω and π_B is an Ω-surjection. If we put $(b,b')(\pi_B\times\pi_B)=(u,u')$, we see

(*) The identity functor on $\mathfrak{A}$.

that there exist $u_0, u_1, \cdots, u_n \in \hat{B}$ such that $u_0 = u$, $u_n = u'$ and $(u_i, u_{i+1}) \in \overline{(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})} \cup \Delta_{\hat{B}}$, $0 \leqq i \leqq n-1$; therefore $(u, u') \in \rho_{\hat{\alpha}}$, and so $\rho_\alpha(\pi_B \times \pi_B) \subset \rho_{\hat{\alpha}}$.

ii) ρ_α is an Ω-subalgebra of B^2 and a congruence on B; π_B is an Ω-surjection; therefore $\rho_\alpha(\pi_B \times \pi_B)$ is an Ω-subalgebra of $\hat{B}^2$ which is reflexive and symmetric; and so the mere application of the operation $\sim$ will be enough to produce an Ω-congruence: but we have just seen that $\rho_\alpha(\pi_B \times \pi_B) \subset \rho_{\hat{\alpha}}$, and then $\overline{\rho_\alpha \widetilde{(\pi_B \times \pi_B)}} \subset \rho_{\hat{\alpha}}$; however

$$(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha}) = (Aa, A\alpha)(\pi_B \times \pi_B) \subset \rho_\alpha(\pi_B \times \pi_B) \subset \overline{\rho_\alpha \widetilde{(\pi_B \times \pi_B)}},$$

and since $\rho_{\hat{\alpha}}$ is, by definition, the smallest Ω-congruence containing $(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})$, we must have $\overline{\rho_\alpha \widetilde{(\pi_B \times \pi_B)}} = \rho_{\hat{\alpha}}$.

iii) Let $(u, u') \in \overline{(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})} \cup \Delta_{\hat{B}}$; then there are three formally distinct possibilities:

a) $(u, u') \in \Delta_{\hat{B}}$: then $u = u'$ and, π_B being a surjection, there exists $b \in B$ such that $b\pi_B = u$; then $(b, b) \in \Delta_B \subset \overline{(A\alpha, A\alpha)} \cup \Delta_B$ and $(b, b)(\pi_B \times \pi_B) = (u, u')$;

b) $(u, u') \in (\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})$: then there exist $b, b' \in A\alpha$ such that $(b, b')(\pi_B \times \pi_B) = (u, u')$ and $(b, b') \in \overline{(A\alpha, A\alpha)} \cup \Delta_B$;

c) (u, u') = composition of elements from $(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})$ with elements from $\Delta_{\hat{B}}$; let us say that $(u, u') = (r_1, r_2)\cdot(s, s)(\omega \times \omega)$, where $r_1, r_2 \in \hat{A}\hat{\alpha}$, $s \in \hat{B}$ and $\omega \in \Omega_2$; then there exist $b_1, b_2 \in A\alpha$ and $b \in B$ such that

$$(b_1, b_2)(\pi_B \times \pi_B) = (r_1, r_2), \qquad b\pi_B = s;$$

then

$$(b_1, b_2)(b, b)(\omega \times \omega) \in \overline{(A\alpha, A\alpha)} \cup \Delta_B$$

and

$$(b_1, b_2)(b, b)(\omega \times \omega)(\pi_B \times \pi_B) = (r_1, r_2)(s, s)(\omega \times \omega).$$

Different cases may be similarly treated.
We are now in a position to state

16. Proposition: $\overline{\mathfrak{U}}$ is right almost exact.

Proof: We consider a saes

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C;$$

in order to simplify the notation we shall write $\overline{\mathfrak{U}}(A) = \hat{A}$ and similarly for B and C, $\overline{\mathfrak{U}}(\alpha) = \hat{\alpha}$ and similarly for β. We then want to show that the sequence

$$\hat{A} \xrightarrow{\hat{\alpha}} \hat{B} \xrightarrow{\hat{\beta}} \hat{C}$$

satisfies

i) $\qquad \hat{\beta}^{-1}\hat{\beta} = \Delta_{\hat{C}}$

ii) $\qquad \rho_{\hat{\alpha}} = \hat{\beta}\,\hat{\beta}^{-1}$.

$$\begin{tikzcd}
B \arrow[r, "\pi_B"] \arrow[d, "\beta"'] & B/\overline{\mathfrak{V}}_B & (=\hat{B}) \\
C \arrow[r, "\pi_C"] & C/\overline{\mathfrak{V}}_C & (=\hat{C})
\arrow[from=1-2, to=2-2, "\hat{\beta}"]
\end{tikzcd}$$

First, since $\overline{\mathfrak{V}}$ is a functor, β induces $\hat{\beta}$ which makes the above diagram commutative; $\hat{\beta}$ has got to be surjective, therefore one has $\hat{\beta}^{-1}\hat{\beta} = \Delta_{\hat{C}}$; next we consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\pi_A"'] & B \arrow[r, "\beta"] \arrow[d, "\pi_B"] & C \arrow[d, "\pi_C"] \\
\hat{A} \arrow[r, "\hat{\alpha}"] & \hat{B} \arrow[r, "\hat{\beta}"] & \hat{C}
\end{tikzcd};$$

here

$$\hat{\beta}\hat{\beta}^{-1} = \pi_B^{-1}\beta\pi_C\pi_C^{-1}\beta^{-1}\pi_B = \pi_B^{-1}\beta\overline{\mathfrak{V}}_C\beta^{-1}\pi_B =$$
$$= \pi_B^{-1}\beta\overline{\widetilde{\overline{\mathfrak{V}}_B(\beta\times\beta)}}\beta^{-1}\pi_B,$$

since $\overline{\mathfrak{V}}$ almost preserves surjections; we shall show that

$$\rho_{\hat{\alpha}} = \pi_B^{-1}\beta\overline{\widetilde{\overline{\mathfrak{V}}_B(\beta\times\beta)}}\beta^{-1}\pi_B;$$

if $(u, u') \in \hat{\beta}\hat{\beta}^{-1}$, then, for some $n \geqq 1$, there exist $b_0, b_n \in B$ such that

$$(b_0, b_n)(\pi_B \times \pi_B) = (u, u'), \qquad (b_0, b_n)(\beta\times\beta) = (c_0, c_n),$$

and there exist $c_1, c_2, \cdots, c_{n-1} \in C$ such that (c_0, c_1), $(c_1, c_2), \cdots$, $(c_{n-1}, c_n) \in \overline{\mathfrak{V}}_B(\beta\times\beta)$; therefore, there exist $b_1, b'_1, b_2, b'_2, \cdots$, $b_{n-1}, b'_{n-1} \in B$ such that (b_0, b_1), $(b'_1, b_2), \cdots, (b'_{n-1}, b_n) \in \overline{\mathfrak{V}}_B$; the action of $\beta\times\beta$ on these elements is described by

$$(b_0, b_1), (b'_1, b_2), \cdots, (b'_{n-1}, b_n) \xrightarrow{\beta\times\beta} (c_0, c_1), (c_1, c_2), \cdots, (c_{n-1}, c_n);$$

but the action of $\pi_B \times \pi_B$ is illustrated by the following diagram

$$\begin{array}{cccccl}
(b_1, b'_1), & (b_2, b'_2), & \cdots, & (b_{n-1}, b'_{n-1}) & \in & \beta\beta^{-1} = \rho_\alpha \\
\downarrow & \downarrow & & \downarrow & & \pi_B \times \pi_B \\
(u_0, u_1), & (u_1, u_2), & \cdots, & (u_{n-2}, u_{n-1}) & \in & \rho_{\hat{\alpha}}
\end{array}$$

and

$$u_0 = b_1\pi_B = b_0\pi_B = u; \qquad u_{n-1} = b'_{n-1}\pi_B = b_n\pi_B = u';$$

therefore $(u, u') \in \rho_{\hat{\alpha}}$ and this shows that $\hat{\beta}\hat{\beta}^{-1} \subset \rho_{\hat{\alpha}}$.

If $(u, u') \in \rho_{\hat{\alpha}}$, then, for some $n \geqq 1$, there exist $u_0, u_1, \cdots, u_n$ such that $u_0 = u$, $u_n = u'$ and $(u_i, u_{i+1}) \in \overline{(\hat{A}\hat{\alpha}, \hat{A}\hat{\alpha})} \cup \Delta_B$; consequently there exist $b_0, b_1, b'_1, b_2, b'_2, \cdots, b'_{n-1}, b_n \in B$ such that

$$\begin{array}{cccccl}
(b_0, b_1), & (b'_1, b_2), & \cdots, & (b'_{n-1}, b_n) & \in & \rho_\alpha = \beta\beta^{-1} \\
\downarrow & \downarrow & & \downarrow & & \pi_B \times \pi_B \\
(u_0, u_1), & (u_1, u_2), & \cdots, & (u_{n-1}, u_n) & ; &
\end{array}$$

but $(b_1, b'_1), (b_2, b'_2), \cdots, (b_{n-1}, b'_{n-1}) \in \overline{\mathfrak{V}}_B$ and

$$(b_1, b'_1),\ (b_2, b'_2), \cdots, (b_{n-1}, b'_{n-1}) \xrightarrow{\beta\times\beta} (c_0, c_1),\ (c_1, c_2), \cdots,$$
$$(c_{n-2}, c_{n-1}) \in \overline{\mathfrak{V}}_B(\beta\times\beta);$$

this means that $(c_0, c_{n-1}) \in \overline{\widetilde{\overline{\mathfrak{V}}_B(\beta\times\beta)}}$; now

$$b_0\beta = b_1\beta = c_0 \quad , \quad b_n\beta = b'_{n-1}\beta = c_{n-1};$$

therefore

$$(u, u') \in \pi_B^{-1}\beta\overline{\widetilde{\overline{\mathfrak{V}}_B(\beta\times\beta)}}\beta^{-1}\pi_B,$$

and so $\rho_{\hat{\alpha}} \subset \hat{\beta}\hat{\beta}^{-1}$. Consequently $\rho_{\hat{\alpha}} = \hat{\beta}\hat{\beta}^{-1}$.

Having shown that $\overline{\mathfrak{U}}$ is right almost exact, one would like to prove that, if $\mathfrak{F}/\mathfrak{g}$ is a $QF\mathfrak{F}$ which is almost right exact, then $\mathfrak{F}/\mathfrak{g} = \overline{\mathfrak{U}}$, or what amounts to the same: $\mathfrak{g} = \overline{\mathfrak{V}}$ for some variety $\mathfrak{V}$. But here we are met with a difficulty which consists in the impossibility of establishing a $1-1$ correspondence between the set of saes of type (2) and the set of Ω-surjections $\beta: B \to C$. It will be convenient to analyse more closely the concept of right almost exactness in the case of $QF\mathfrak{F}$'s.

Given a $QF\mathfrak{F}$, which we denote by $(\hat{-})$ for brevity, it will be right almost exact if, given any saes

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

then

$$\hat{A} \xrightarrow{\hat{\alpha}} \hat{B} \xrightarrow{\hat{\beta}} \hat{C}$$

is right almost exact, i. e., is such that $\hat{\beta}^{-1}\hat{\beta}=\Delta_{\hat{C}}$, $\rho_{\hat{\alpha}}=\hat{\beta}\hat{\beta}^{-1}$. We remark three things. First, that we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_C} \\ \hat{A} & \xrightarrow{\hat{\alpha}} & \hat{B} & \xrightarrow{\hat{\beta}} & \hat{C} \end{array}.$$

Second, that the condition $\hat{\beta}^{-1}\hat{\beta}=\Delta_{\hat{C}}$, which expresses the surjectivity of $\hat{\beta}$, is automatically satisfied in the case of a $QF\vartheta$; therefore, the relevant condition is $\rho_{\hat{\alpha}}=\hat{\beta}\hat{\beta}^{-1}$. Third, that in the present circumstances (cf. 15. Lemma) $\rho_{\hat{\alpha}}=\overline{\rho_\alpha\widetilde{(\pi_B\times\pi_B)}}$; and this means that the condition $\rho_{\hat{\alpha}}=\hat{\beta}\hat{\beta}^{-1}$ may be expressed as

$$\hat{\beta}\hat{\beta}^{-1}=\overline{\beta\beta^{-1}\widetilde{(\pi_B\times\pi_B)}}$$

or, if we put

$$\beta\beta^{-1}(\pi_B\times\pi_B)=\widehat{\beta\beta}^{-1},$$

as

$$\hat{\beta}\hat{\beta}^{-1}=\widetilde{\widehat{\beta\beta}}^{-1}.$$

This condition may be expressed by saying that the functor $(\hat{\;})$ *almost preserved the* Ω*-congruence*.

We shall, then, say that a $QF\vartheta$ *almost preserves* Ω*-congruences* if, whenever we are given an Ω-surjection $\beta: B\to C$, then, in the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \pi_C} \\ \hat{B} & \longrightarrow & \hat{C} \end{array},$$

we have

$$\hat{\beta}\hat{\beta}^{-1}=\widetilde{\widehat{\beta\beta}}^{-1}=\overline{\beta\beta^{-1}\widetilde{(\pi_B\times\pi_B)}}=\bigcup_{n\geqq 1}^{\infty}(\beta\beta^{-1}(\pi_B\times\pi_B))^n.$$

Obviously, such a functor is right almost exact. Now we can prove

17. Proposition: If a $QF\vartheta$, ϑ/g, almost preserves Ω-congruences, then $\vartheta/g=\overline{\mathfrak{V}}$, for some variety.

Proof: From the diagram above we see that, in any case,

$$\hat{\beta}\hat{\beta}^{-1}=\pi_B^{-1}\beta g(C)\beta^{-1}\pi_B.$$

Now, $(u,u')\in\overline{\beta\beta^{-1}\widetilde{(\pi_B\times\pi_B)}}=\hat{\beta}\hat{\beta}^{-1}$ implies that, for some $n\geqq 1$, there exist $u_0, u_1, \cdots, u_n\in\hat{B}$ such that $u_0=u$, $u_n=u'$ and $(u_i, u_{i+1})\in\beta\beta^{-1}(\pi_B\times\pi_B)$; therefore, there exist $b_0, b_1, b'_1, b_2, b'_2, \cdots \cdots, b'_{n-1}, b_n\in B$ such that

$$\begin{array}{ccccccc} (b_0, b_1), & (b'_1, b_2), & \cdots, & (b'_{n-1}, b_n) & \in & \beta\beta^{-1} & \\ \downarrow & \downarrow & & \downarrow & & & \pi_B\times\pi_B \\ (u_0, u_1), & (u_1, u_2), & \cdots, & (u_{n-1}, u_n) & & & \end{array}$$

$$(b_1, b'_1), \cdots, (b_{n-1}, b'_{n-1})\in g(B)\xrightarrow{\beta\times\beta}(c_0, c_1), \cdots,$$
$$(c_{n-2}, c_{n-1})\in g(B)(\beta\times\beta);$$

therefore

$$(c_0, c_{n-1})\in\overline{g\widetilde{(B)(\beta\times\beta)}},$$

and so

$$(u, u')\in\pi_B^{-1}\beta\,\overline{g\widetilde{(B)(\beta\times\beta)}}\beta^{-1}\pi_B;$$

this shows that

$$\hat{\beta}\hat{\beta}^{-1}=\pi_B^{-1}\beta g(C)\beta^{-1}\pi_B\subset\pi_B^{-1}\beta\overline{\widetilde{g(B)(\beta\times\beta)}}\beta^{-1}\pi_B,$$

and so we must have

$$g(C)\subset\overline{\widetilde{g(B)(\beta\times\beta)}},$$

hence

$$g(C)=\overline{\widetilde{g(B)(\beta\times\beta)}};$$

but this tells us that $g(-)$ almost preserves surjections, and so $g(-)=\mathfrak{V}(-)$, for some variety $\mathfrak{V}$.

We may collect our results in 2. Theorem.

In the case of groups with multiple operators or Ω-groups one may obtain a stricter version of this theorem. An Ω-group G [5] is a set such that

G. 1. G is a group (not necessarily commutative) with respect to the operators $(+)$ and $(-)$;

G. 2. G admits the set Ω of finitary operators;

G. 3. for every $\omega\in\Omega$, $00\cdots0\omega=0$, where 0 is the zero element of G.

One defines the concept of ideal of an Ω-group, which is a generalization of that of normal subgroup of an ordinary group and of that of ideal of a ring. Given an Ω-homomorphism $\varphi:A\to B$ one defines $\ker\varphi=\{x\in A\mid x\varphi=0\}$: $\ker\varphi$ is an ideal of A; conversely, given any ideal L of A, then $\ker(\psi:A\to A/L)=L$. There is a $1-1$ correspondence between ideals and Ω-congruences.

We consider a category $\mathfrak{C}$ of Ω-groups (Ω fixed) and a variety $\mathfrak{V}\subset\mathfrak{C}$; we consider, moreover, the associated functors $\overline{\mathfrak{V}}(-)$ and $\overline{\mathfrak{U}}(-)$. For every Ω-group A we can associate with $\overline{\mathfrak{V}}(A)$ the homomorphism $\alpha:A\to A/\overline{\mathfrak{V}}(A)$. Let us put $V(A)=\ker\alpha$. Then $V(A)$ is the unique ideal of A which generates $\overline{\mathfrak{V}}(A)$. We shall see that $V(-)$ is a functor $V:\mathfrak{C}\to\mathfrak{C}$ which is a subfunctor of the identity functor $\mathfrak{I}$ in $\mathfrak{C}$, and that we may associate it with the variety $\mathfrak{V}$. For every $A\in\mathfrak{C}$, if we denote by α_{ij} all the morphisms

$A\to A_i$ where A_i runs through the elements of $\mathfrak{V}$, then $V(A)$ is explicitely of the form $V(A)=\bigcap_{ij}\ker\alpha_{ij}$. We put $U(A)=A/$ $/V(A)=A/\overline{\mathfrak{V}}(A)=\overline{\mathfrak{U}}(A)$. Then we can state 1. Theorem.

Proof: The proof of this theorem may be found in [3]; here, however, we shall show how to derive it from our previous results.

i) $U(A)=\overline{\mathfrak{U}}(A)$, for every $A\in\mathfrak{C}$, therefore $U(A)\in\mathfrak{V}$, for every $A\in\mathfrak{C}$, by 2. Theorem; moreover, we know that $A\in\mathfrak{V}$ if and only if $\overline{\mathfrak{V}}(A)=\Delta_A$; but

$$\overline{\mathfrak{V}}(A)=\overline{\widetilde{(V(A),V(A))\cup\Delta_A}}=\Delta_A\Longleftrightarrow V(A)=0\Longleftrightarrow U(A)=A.$$

ii) In order to show that $V(-)$ is a functor we essentially have to show that, given $\varphi:A\to B$, then $V(\varphi)=\varphi\mid V(A):V(A)\to V(B)$. But we know that φ induces a homomorphism

$$\overline{\mathfrak{V}}(A)=\overline{\widetilde{(V(A),V(A))\cup\Delta_A}}\to\overline{\mathfrak{V}}(B)=\overline{\widetilde{(V(B),V(B))\cup\Delta_B}},$$

this map being a restriction of $\varphi\times\varphi$; moreover

$$\overline{\widetilde{(V(A)\varphi,V(A)\varphi)\cup\Delta_B}}\subset\overline{\widetilde{(V(B),V(B))\cup\Delta_B}}$$

since, in fact, $(a_1,a_2)\in\overline{\mathfrak{V}}(A)$ for every $a_1,a_2\in V(A)$, and therefore $(a_1\varphi,a_2\varphi)\in\overline{\mathfrak{V}}(B)$, which means that the Ω-congruence on B generated by $(V(A)\varphi,V(A)\varphi)$ is contained in $\overline{\mathfrak{V}}(B)$. From the previous inclusion we deduce $V(A)\varphi\subset V(B)$.

$V(A)$ is an ideal of A, by definition, and it is not difficult to see that $V(A)=\bigcap_{A_i\in\mathfrak{V}}\ker\alpha_{ij}$ for, in fact,

$$\overline{\mathfrak{V}}(A)=\bigcap\{F_i\in\mathfrak{C}_\Omega(A)\mid A/F_i\in\mathfrak{V}\}.$$

For every projection $\alpha_{io}:A\to A/F_i$ one has

$$\overline{\widetilde{(\ker\alpha_{io},\ker\alpha_{io})\cup\Delta_A}}=F_i$$

and so $V(A)\subset\ker\alpha_{io}$ for every io, therefore $V(A)\subset\bigcap_{A_i\in\mathfrak{V}}\ker\alpha_{ij}$; but $A/V(A)=A/\overline{\mathfrak{V}}(A)\in\mathfrak{V}$ ond so $V(A)=\bigcap_{A_i\in\mathfrak{V}}\ker\alpha_{ij}$.

Now, given a surjection $\varphi : A \to B$, let us consider $V(A)$ and $V(B)$. In the circumstances $V(A)\varphi \lhd B$(*). Since $\overline{\mathfrak{V}}(-)$ almost preserves surjections, one has

$$\overline{\widetilde{\overline{(V(A), V(A))\cup\Delta_A}}}(\varphi\times\varphi) = \overline{\widetilde{\overline{(V(B), V(B))\cup\Delta_B}}};$$

however

$$\overline{\widetilde{\overline{(V(A), V(A))\cup\Delta_A}}}(\varphi\times\varphi) \subset \overline{\widetilde{\overline{(V(A)\varphi, V(A)\varphi)\cup\Delta_B}}},$$

and so

$$\overline{\widetilde{\overline{(V(B), V(B))\cup\Delta_B}}} \subset \overline{\widetilde{\overline{(V(A)\varphi, V(A)\varphi)\cup\Delta_B}}},$$

which means that $V(B) \subset V(A)\varphi$; therefore $V(A)\varphi = V(B)$, i. e., V preserves surjections.

Conversely, if $T \lhd \mathfrak{I}$ is a functor which preserves surjections, then given a surjection $\varphi : A \to B$, one will have $T(B) = T(A)\varphi$.

If we define $\mathfrak{T}$ by $\mathfrak{T}(A) = \overline{\widetilde{\overline{(T(A), T(A))\cup\Delta_A}}}$ we shall have

$$\overline{\widetilde{\overline{(T(A), T(A))\cup\Delta_A}}}(\varphi\times\varphi) = \overline{\widetilde{\overline{(T(A)\varphi, T(A)\varphi)\cup\Delta_B}}} = \\ = \overline{\widetilde{\overline{(T(B), T(B))\cup\Delta_B}}}$$

since $\Delta_A(\varphi\times\varphi) = \Delta_B$. This means that $\mathfrak{T}$ almost preserves surjections, therefore $\mathfrak{T}(-) = \mathfrak{V}(-)$, for some variety $\mathfrak{V}$, therefore $T = V$ for the same variety $\mathfrak{V}$.

iii) The proof does not differ from the one given in |3| since $U = \overline{\mathfrak{U}}$.

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(*) $\lhd$ means «is an ideal of».

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BIBLIOGRAPHY

1. CARTAN, H. & EILENBERG, S. *Homological algebra*, Princeton Univ. Press, Princeton, 1956.

2. COHN, P. M. *Universal algebra*, Harper & Row, New York and London, 1965.

3. FRÖHLICH, A. *On groups over a d. g. near-ring III. Categories and functors*; Quart. J. Math. Oxford Ser. (2) **2** (1960), 211-228.

4. FURTADO-COELHO, J. *Varieties of Ω-groups and associated functors*, Ph. D. thesis, University of London, 1972.

5. HIGGINS, P. J. *Groups with multiple operators*, Proc. London Math Soc. (3) **6** (1956), 366-416.

6. HIGGINS, P. J. *Lecture notes of a course on universal algebras delivered at King's College London in 1970-1971* (unpublished).

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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UNE PROPRIÉTÉ CARACTÉRISTIQUE D'UN IDÉAL À GAUCHE $\mathscr{P}$-ISOTOPIQUE DANS UN ANNEAU NOETHÉRIEN À GAUCHE (*)

PAR

L. LESIEUR

En hommage à M. le Professeur A. ALMEIDA COSTA

Nous donnons dans cet article une propriété caractérisant, d'une façon interne à l'anneau, un idéal à gauche $\mathscr{P}$-isotypique, où $\mathscr{P}$ désigne un idéal bilatère premier d'un anneau R noethérien à gauche.

1. ENONCÉ DU THÉORÈME

Soit $\mathscr{P}$ un idéal bilatère premier d'un anneau R noethérien à gauche unitaire. On sait que l'enveloppe injective $E(R/\mathscr{P})$, en tant que R-module à gauche, est la somme directe de n modules injectifs indécomposables isomorphes : $E(R/\mathscr{P}) = E_1 \oplus \cdots \oplus E_n$.

Définition 1. Un idéal à gauche Q de R est *$\mathscr{P}$-isotypique* si l'enveloppe injective $E(R/Q)$ est somme directe d'un nombre fini de modules injectifs indécomposables isomorphes à E_1.

Ainsi, $\mathscr{P}$ lui-même est un idéal $\mathscr{P}$-isotypique. Dans le cas commutatif, un idéal $\mathscr{P}$-isotypique coïncide avec un idéal $\mathscr{P}$-primaire. Nous allons donner une propriété caractéristique nouvelle d'un idéal $\mathscr{P}$-isotypique, interne à l'anneau R.

(*) Recebido em 9 de Dezembro de 1973.

Théorème 1. 1. *Pour que l'idéal à gauche* $Q \neq R$ *soit* $\mathscr{P}$*-isotypique, il faut et il suffit qu'il vérifie la condition suivante :*

(C) $\quad \forall b \notin Q, \ \exists \lambda_i \in R, \ i=1, \cdots, n$ *tels que* $\mathscr{P} = \bigcap_{i=1}^{i=n} (Q \cdot . \lambda_i b)$.

Dans cet énoncé, $Q \cdot . a = \{x \in R \mid x a \in Q\}$.
Pour la démonstration, on utilise le lemme suivant.

2. Lemme 2. 1. *Si* $Q \neq R$ *est un idéal* $\mathscr{P}$*-isotypique contenant* $\mathscr{P}$, *et si* $\mathscr{C}(\mathscr{P})$ *désigne l'ensemble des éléments réguliers mod.* $\mathscr{P}$, *on a :*

$$Q \cap \mathscr{C}(\mathscr{P}) = \phi .$$

En effet, on sait que $Q = \mathrm{Ann}\, \beta$ est l'annulateur d'un élément $\beta \neq 0$ de l'enveloppe injective $E(R/\mathscr{P})$. On a donc, $E(R/\mathscr{P})$ étant extension essentielle de $R/\mathscr{P}$,

$$0 \neq \lambda \beta = \bar{a} \in R/\mathscr{P}, \ \lambda \in R, \ a \in R, \ a \notin \mathscr{P}.$$

Supposons qu'il existe $c \in \mathscr{C}(\mathscr{P}) \cap Q$. On aurait donc : $c\beta = 0$. Appliquons la condition de Ore dans $R/\mathscr{P}$ à c et λ, ce qui donne dans R : $\sigma\lambda = tc + p$, $\sigma \in \mathscr{C}(\mathscr{P})$, $t \in R$, $p \in \mathscr{P}$. Comme $Q\beta = 0$ et que $\mathscr{P} \subseteq Q$, on a :

$$\sigma\lambda\beta = tc\beta + p\beta = 0 \Rightarrow \sigma\bar{a} = 0 \Rightarrow \sigma a \in \mathscr{P}.$$

Mais $\sigma \in \mathscr{C}(\mathscr{P})$, d'où $a \in \mathscr{P}$ (contradiction).

3. DÉMONSTRATION DU THÉORÈME. CONDITION NÉCESSAIRE

Soit Q un idéal à gauche $\mathscr{P}$-isotypique. Décomposons $Q = Q_1 \cap \cdots \cap Q_m$ en intersection d'idéaux à gauche Q_p non superflus $\cap$-irréductibles. Ces idéaux à gauche $Q_p (p = 1, \cdots, m)$ sont également $\mathscr{P}$-isotypiques. Soit $b \notin Q$; comme $\mathscr{P} = \mathrm{Ass}\, Q$, il existe un idéal à gauche Z de R tel que $Zb \not\subseteq Q$ et $\mathscr{P} = Q \cdot . Zb = \bigcap_{z \in Z} (Q \cdot . zb)$, d'où :

$$\mathscr{P} = \Big(\bigcap_{z \in Z} Q_1 \cdot . zb)\Big) \cap \cdots \cap \Big(\bigcap_{z \in Z} (Q_m \cdot . zb)\Big).$$

Mais $Q_p \cdot . zb$ est, pour tout z, un ideal à gauche Q' $\cap$-irréductible, $\mathscr{P}$-isotypique, contenant $\mathscr{P}$. Il vérifie donc d'après le lemme : $Q' \cap \mathscr{C}(\mathscr{P}) = \phi$, et il en résulte, d'après un résultat de Lambek et Michler [1], que Q' est $\mathscr{P}$-critique. $\mathscr{P}$ est égal à une intersection d'idéaux à gauche $\mathscr{P}$-critiques ; une telle intersection se réduit alors à une intersection finie. La condition (C) en résulte aisément.

4. DÉMONSTRATION DU THÉORÈME. CONDITION SUFFISANTE

Soit Q un idéal à gauche vérifiant la condition (C). Prenons comme plus haut $Q = Q_1 \cap \cdots \cap Q_m$. Nous allons prouver que chaque composante Q_p est $\mathscr{P}$-isotypique. Il existe $b \in Q_2 \cap \cdots \cap Q_m$, $b \notin Q_1$, donc $b \notin Q$. D'après la condition (C) :

$$\mathscr{P} = \bigcap_{i=1}^{n} (Q \cdot . \lambda_i b) = \bigcap_{i=1}^{n} (Q_1 \cdot . \lambda_i b).$$

En gardant dans cette dernière intersection les composantes non superflues il reste des idéaux à gauche $\cap$-irréductibles qui sont nécessairement $\mathscr{P}$-critiques, donc $\mathscr{P}$-isotypiques. Q_1, étant isotypique du même type que $Q_1 \cdot . \lambda_i b$, est également $\mathscr{P}$-isotypique. On établit le même résultat pour $Q_2, \cdots, Q_m$, et par suite pour Q.

5. UNE APPLICATION

On peut espérer que la condition (C) puisse jouer, dans le cas non commutatif, le même rôle que celle qui définit un idéal $\mathscr{P}$-primaire dans le cas commutatif. On en trouvera un exemple dans [2] pour démontrer que, si R possède un anneau de fractions classiques R_S pour $S = \mathscr{C}(\mathscr{P})$, il y a une correspondance biunivoque canonique entre les idéaux à gauche $\mathscr{P}$-isotypiques de R et les idéaux à gauche $e(\mathscr{P})$-isotypiques de R_S, où $e(\mathscr{P})$ désigne l'idéal bilatère premier $R_S\mathscr{P}$ qui est l'extension de $\mathscr{P}$ dans R_S (cf. propriété 15).

BIBLIOGRAPHIE

[1] J. LAMBEK and G. MICHLER, *The Torsion Theory at a Prime Ideal of a Right Noetherian Ring* (Journal of Algebra, Vol. 25, n.º 2, 1973, pp. 364-390).

[2] L. LESIEUR, *Idéal premier $\mathcal{P}$ d'un anneau noethérien à gauche* (Séminaire d'Algèbre non commutative, Publications Mathématiques d'Orsay, 1973, conférence n.º 3).

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SOBRE BI-IDEALES DE SEMIANILLOS(*)

POR

FRANCISCO POYATOS

Homenaje al Prof. ALMEIDA COSTA

El objeto del presente artículo consiste en establecer aquellas proposiciones en las que nos apoyaremos para demostrar los teoremas de GREEN y JORDAN-HÖLDER en semianillos. Esto último lo haremos en otro artículo que publicaremos próximamente bajo el título «Series principales y series de composición de semianillos». En ambos artículos seguimos principalmente los trabajos que sobre semigrupos han realizado CLIFFORD y PRESTON [1], GREEN [4] y REES [9]. Ambos artículos son paralelos a [7].

1. INTRODUCCIÓN

Entendemos aquí por semianillo un conjunto $A \neq \phi$ dotado de dos operaciones, que designamos «$+$» y «$\cdot$» (o yuxtaposición), que llamamos adición y multiplicación, de modo que se cumple:

1) $(A, +)$ es un semigrupo conmutativo;

2) $(A, \cdot)$ es un semigrupo;

3) distributividad de la multiplicación respecto de la adición a izquierda y derecha.

Decimos que el semianillo A es con cero ó que tiene cero si y sólo se cumple 4) y 5):

(*) Recebido em 18 de Dezembro de 1973.

4) existe un elemento neutro respecto de la adición en A, que designamos «0», y tal que verifica

5) $a \cdot 0 = 0 \cdot a = 0, \quad \forall a \in A$.

Llamamos al semianillo unitario ó con elemento unidad si y solo si

6) existe en A un elemento neutro respecto de la multiplicación, que designamos «1».

Cualquier semianillo A que no contenga elemento neutro respecto de la adición, ó que, conteniéndolo, sea «u», u no satisfaga la condición 5), puede ser sumergido de una manera trivial en un semianillo $A_0 = A \cup \{0\}$, $0 \notin A$, en el que «0» cumple 4) y 5). Basta definir las operaciones en A_0 de este modo:

$$x + y \text{ en } A_0 = x + y \text{ en } A, \quad \forall x, y \in A.$$

$$x + 0 = 0 + x = 0, \quad \forall x \in A.$$

$$0 + 0 = 0.$$

$$x \cdot y \text{ en } A_0 = x \cdot y \text{ en } A, \quad \forall x, y \in A.$$

$$x \cdot 0 = 0 \cdot x = 0, \quad \forall x \in A.$$

$$0 \cdot 0 = 0.$$

Un semianillo A que satisfaga 4) y 5) y no contenga elemento unidad puede ser sumergido en otro semianillo que llamamos $(A)_1$ definido de esta manera:

$(A)_1 = A \times N$, siendo N el semianillo de los enteros no negativos, dotado de las operaciones:

$$(a, m) + (b, n) = (a + b, m + n),$$

$$(a, m) \cdot (b, n) = (a \cdot b + n a + m b, m \cdot n),$$

en el cual $(0, 0)$ es el cero y $(0, 1)$ es el elemento unidad. Además $M = \{(a, 0)\}_{a \in A}$ es subsemianillo de $(A)_1$ isomorfo a A.

Si A contiene elemento unidad pero no elemento cero, entonces basta efectuar la primera inmersión (con más detalle puede verse en [6]).

Convenimos en que si B, C, son partes del semianillo A y c es un elemento de A, entonces

$$B + C = \{b + c \in A / b \in B, c \in C\},$$

$$B \cdot C = \{b \cdot c \in A / b \in B, c \in C\} = B C,$$

$$B + c = B + \{c\},$$

$$B \cdot c = B \cdot \{c\} = B c.$$

Decimos que $D \subseteq A$, es subsemianillo del semianillo A si y solo si D satisface:

7) $D + D \subseteq D$,

8) $D \cdot D \subseteq D$.

Para que D sea subsemianillo de A no exigimos que D contenga al cero, sino sólo que cumpla 7) y 8). Consideramos al vacío $\emptyset$ subsemianillo de cualquier semianillo, que llamamos trivial.

Si el semianillo A es con cero, entonces denominamos $A' = A - \{0\}$; si A no contuviera cero, $A' = A$. (— significa aquí complemento).

Se dirá que $\mathfrak{a} \subseteq A$ es *bi-ideal* del semianillo A si y sólo si $\mathfrak{a}$ verifica:

9) $0 \notin \mathfrak{a}$

10) $\mathfrak{a} + A \subseteq \mathfrak{a}$

11) $A' \cdot \mathfrak{a} \subseteq \mathfrak{a}$

12) $\mathfrak{a} \cdot A' \subseteq \mathfrak{a}$.

Si A no tuviera cero, la condición 9) sería trivialmente verdadera. Es evidente que todo *bi-ideal* del semianillo A es subsemianillo de A. Consideramos a $\emptyset$ también como bi-ideal de A, que llamamos *trivial*. La razón de que denominemos a $\mathfrak{a}$ bi-ideal de A consiste en que $\mathfrak{a}$ es ideal de A respecto de las dos operaciones definidas en A. Es evidente también que todo bi-ideal de A es ideal (sin cero) del *semianillo* A. Szász y Lajos han estudiado lo que ellos llaman «bi-ideals» y «bi-ideales generalizados» en anillos ([5] y [10]), designando con estas palabras conceptos diferentes del nuestro. El vocablo «bi-

-ideal» también ha sido usado en la teoría de semigrupos con significados distintos, como es natural, del nuestro. Así, por ejemplo, en [3], [1] y [2].

Diremos de un semianillo A que es *bi-ideal* (con abuso del lenguaje) si y sólo si A' es bi-ideal de A, esto es si y sólo si:

13) $A' + A \subseteq A'$

14) $A' \cdot A' \subseteq A'$.

La condición 13) equivale a esta otra:

15) $A' + A' \subseteq A'$.

Por tanto, decir del semianillo A que es bi-ideal equivale a decir que A' es subsemianillo de A. En tal caso A' es el máximo bi-idéal de A. La condición 15) indica que A es sin opuestos (es decir, que no existe ningún $z \neq 0$ en A tal que haya un $z' \in A$ de modo que $z + z' = 0$). La condición 14) significa que A es sin divisores de cero.

Llamamos a $w \in A$ *elemento distinguido* del semianillo A si cumple

16) $w + x = w, \quad \forall x \in A,$

17) $a' \cdot w = w \cdot a' = w, \quad \forall a' \in A'.$

Caso de que el semianillo A tenga un elemento distinguído, éste es único. En el semianillo $O = \{0\}$ con las leyes de composición:

$$0 + 0 = 0, \qquad 0 \cdot 0 = 0$$

el cero es también elemento distinguido, lo que expresamos $O = \{w\} = W$. En cualquier otro semianillo con cero, con elemento distinguido y con, al menos, un elemento diferente de ambos es $w \neq 0$, debido a las condiciones 4) y 16). w designará siempre en lo sucesivo un elemento distinguido de un semianillo, y a aquellos semianillos que contengan un elemento distinguido w les llamaremos, para abreviar, w-semianillos. Además W designará siempre $W = \{w\}$.

El semianillo $O_w = \{0, 1\}$, $0 \neq 1$, en el que 0 es el cero y 1 es el elemento unidad y en donde se verifica

$$1 + 1 = 1$$

tiene por elemento distinguido $w = 1$. Por ello también lo designamos $O_w = \{0, w\}$. En cualquier otro w-semianillo con cero y unitario es $w \neq 1$, debido a las condiciones 6) y 17).

Dado el bi-ideal $\mathsf{a} \neq \emptyset$ del semianillo A construimos la relación binaria en A:

$$x R_{\mathsf{a}} y \iff (x = y) \vee (x, y \in \mathsf{a}), \quad (\vee \text{ , disyunción lógica}),$$

la cual es una congruencia en A (esto es, una relación de equivalencia en A compatible con sus dos operaciones), que llamamos «congruencia de Rees en A engendrada por a», y denominamos así por analogía con las congruencias definidas por Rees en los semigrupos no conmutativos [9] y con las congruencias de Rees en A-semimódulos a la izquierda [7]. Definimos A/a, que llamamos semianillo cociente de Rees, de este modo:

$$A/\mathsf{a} = A/R_{\mathsf{a}} = (A - \mathsf{a}) \cup \{w\},$$

siendo «$-$» complemento y «w» elemento distinguido de A/R_{a}. Todos los cocientes tratados en este artículo serán cocientes de Rees respecto de bi-ideales.

2. PROPIEDADES DE LOS BI-IDEALES

Recordemos que el vacío $\emptyset$ es subsemianillo y bi ideal, denominado trivial, de cualquier semianillo A. El lector comprobará la

Proposición 1

i) *La intersección de una familia cualquiera no vacía de bi-ideales de un semianillo* A *es un bi-ideal del semianillo.*

ii) *La reunión de una familia cualquiera de bi-ideales de* A *es un bi-ideal de* A.

iii) *Si* b *y* c *son bi-ideales no triviales de* A, *entonces* $\mathsf{b}+\mathsf{c}$ *es bi-ideal no trivial de* A *tal que*

$$\mathsf{b} + \mathsf{c} \subseteq \mathsf{b} \cap \mathsf{c}.$$

DEFINICIÓN

Sea A un semianillo, llamamos *bi-ideal nuclear de* A, que designamos $\mathbf{k}(A)$, a la intersección, si no es vacía, de todos los bi-ideales no triviales de A. Está claro que, en virtud de la proposición 1 i), $\mathbf{k}(A)$ es bi-ideal de A, como su nombre indica, el cual constituye, por definición, el mínimo bi-ideal no trivial de A, caso de que exista.

Es posible que el semianillo A tenga bi-ideales no triviales y que la intersección de todos ellos sea vacía; en tal caso A carece de bi-ideal nuclear. Así ocurre con el semianillo de los enteros no negativos N. Para que una parte propia T de N sea bi-ideal es necesario y suficiente que exista un $p \in N'$ tal que $T = \{x \in N / x \geq p\}$. Llamemos T_p a T. Como $\bigcap_{p \in N'} T_p = \emptyset$, N carece de bi-ideal nuclear. Por otra parte, todo w-semianillo A es tal que $\mathbf{k}(A) = \{w\} = W$.

El semianillo nulo O es bi-ideal porque $O' = \emptyset$ satisface las condiciones 13) y 14). Asímismo todo semianillo A sin cero es bi-ideal puesto que $A' = A$ cumple también las condiciones 13) y 14). Además

PROPOSICIÓN 2

La condición necesaria y suficiente para que un semianillo con cero A *contenga, al menos, un bi-ideal no trivial* $\mathbf{b}$ *es que* A *sea semianillo bi-ideal diferente de* O.

Demostración. Supongamos que A es semianillo que contiene un cero y un bi-ideal $\mathbf{b} \neq \emptyset$, entonces

$$(A' + A') \cdot \mathbf{b} \subseteq A' \cdot \mathbf{b} + A' \cdot \mathbf{b} \subseteq \mathbf{b} + \mathbf{b} \subseteq \mathbf{b} + A \subseteq \mathbf{b},$$
$$(A' \cdot A') \cdot \mathbf{b} = A' \cdot (A' \cdot \mathbf{b}) \subseteq A' \cdot \mathbf{b} \subseteq \mathbf{b}.$$

Como $0 \notin \mathbf{b} \neq \emptyset$, ha de ser

$$A' \neq \emptyset, \quad A' + A' \subseteq A', \quad A' \cdot A' \subseteq A',$$

es decir, se cumplen las condiciones 15) y 14), o, sus equivalentes, 13) y 14), por lo que A es semianillo con cero en el que A' es bi-ideal no trivial, es decir, A es semianillo con cero bi-ideal diferente de O.

La recíproca es inmediata, c. q. d.

En todo lo que sigue, salvo se indique lo contrario, nos referiremos a semianillos bi-ideales.

PROPOSICIÓN 3

Se puede adjuntar a todo semianillo bi-ideal A *un elemento* w $\notin$ A *de modo que* $A_w = A \cup \{w\}$ *sea* w-*semianillo.*

En efecto, ampliamos las leyes de composición de A a A_w de esta manera:

$$x + w = w = w + x, \qquad \forall\, x \in A,$$
$$w + w = w,$$
$$a' \cdot w = w = w \cdot a', \qquad \forall\, a' \in A'$$
$$w \cdot w = w,$$
$$0 \cdot w = 0 = w \cdot 0, \qquad \text{caso de que } A \text{ contenga un cero.}$$

Entonces A_w es un semianillo con elemento distinguido w. Si A ya tuviera un elemento distinguido, llamémosle w_0, en A_w dejaría w_0 de ser elemento distinguido, para pasar a serlo w. El proceso indicado en la proposición 3 puede reiterarse indefinidamente (véase [7], comentario a la proposición 3).

PROPOSICIÓN 4

Dado el semianillo bi-ideal con cero y unitario A, *para todo* $m \in A'$.

$$J(m) = A' m A' + A$$

es el mínimo bi-ideal de A *que contiene a* m.

Demostración. Si $m \in A'$, entonces

$$A' m \subseteq A' \cdot A' \subseteq A', \qquad A' m A' \subseteq A',$$

por lo que

$$J(m) = A' m A' + A \subseteq A' + A \subseteq A';$$

a fortiori,

$$0 \notin J(m);$$

además

$$J(m) + A = A' m A' + A + A \subseteq A' m A' + A = J(m),$$
$$A' \cdot J(m) = A'(A' m A' + A) \subseteq A' m A' + A = J(m),$$

análogamente

$$J(m) \cdot A' \subseteq J(m);$$

de todo lo cual resulta que $J(m)$ es un bi-ideal de A que, evidentemente, contiene a m.

Sea $\mathbf{b}$ un bi-ideal de A tal que $m \in \mathbf{b}$, entonces

$$A' m \subseteq A' \mathbf{b} \subseteq \mathbf{b}, \quad A' m A' \subseteq \mathbf{b} A' \subseteq \mathbf{b},$$

$$m \in J(m) = A' m A' + A \subseteq \mathbf{b} + A \subseteq \mathbf{b};$$

por lo que $J(m)$ es el mínimo bi-ideal de A que contiene a $m \neq 0$, c. q. d.

3. *R*-SIMPLICIDAD DE SEMIANILLOS Y MINIMALIDAD DE BI-IDEALES

Definiciones

Sea A semianillo; llamamos bi-ideales *impropios* de A a A', W (caso de que existan) y a ϕ. Cualquier otro bi-ideal de A será llamado *propio*.

Un w-semianillo diferente de O es semianillo bi-ideal puesto que todo semianillo sin cero es bi-ideal y todo semianillo con cero y con un bi-ideal no trivial $W = \{w\}$ es, según la proposición 2, semianillo bi-ideal.

Sea A un w-semianillo, se dice que A es *w-simétrico* si se verifica $A' \neq W$ y

$$A' + A' = W \quad \text{ó} \quad A' \cdot A' = W.$$

Se dirá que es *w-simétrico primitivo* si y sólo si a A es w-simétrico y no contiene ningún bi-ideal propio.

Decimos de un semianillo A que es *R-simple* (R —, de Rees) si y sólo si

1.° A no es w-simétrico,

y

2.° A no contiene ningún bi-ideal propio.

Vamos a poner un ejemplo de semianillo conmutativo w-simétrico. Sea A_I el conjunto

$$A_I = \bigcup_{i \in I} \{x_i\} \cup \{0\} \cup \{1\} \cup \{w\},$$

donde todos los elementos son diferentes entre sí dos a dos, y en el que imponemos las leyes de composicón:

$$x + y = w, \quad \forall x, y \in A'_I = A_I - \{0\},$$
$$x + 0 = 0 + x = x, \quad \forall x \in A_I,$$
$$1 \cdot x = x \cdot 1 = x, \quad \forall x \in A_I,$$
$$0 \cdot x = x \cdot 0 = 0, \quad \forall x \in A_I,$$
$$x \cdot y = w, \quad \forall x, y \in A''_I = A_I - \{0, 1\}.$$

El lector comprobará que A_I es un w-semianillo en el que

$$A'_I + A'_I = W,$$

por lo que es w-simétrico. (A'_I es bi-ideal de A_I).

Se $J \subseteq I$, llamemos

$$A_J = \bigcup_{i \in J} \{x_i\} \cup \{0\} \cup \{1\} \cup \{w\},$$
$$A''_J = A_J - \{0, 1\};$$

A''_J, para toda parte de J de I es bi-ideal de A_I, ya que

$$A''_J + A_I = A''_J,$$

$$A''_J \cdot A'_I = A'_I \cdot A''_J = A''_J.$$

A_ϕ (es decir, cuando $I=\phi$) es un semianillo w-simétrico primitivo, puesto que sus únicos bi-ideales no triviales son

$$A'_\phi = \{1, w\}, \quad A''_\phi = \{w\} = W.$$

En cambio, el semianillo $O_w = \{0, w\}$ no es w-simétrico porque $O'_w = W$, y como no contiene ningún bi-ideal propio, es R-simple.

Todo semianillo tiene un bi-ideal mínimo que es el trivial ϕ. Excluyendo éste, para que exista bi-ideal (no trivial) minimal en un semianillo A es necesario que A sea bi-ideal $\neq 0$ (proposición 2). Distinguimos dos casos, según que el semianillo bi-ideal $A \neq 0$ carezca o posea elemento distinguido w. Si A no contiene elemento distinguido entonces pueden suceder dos cosas, que en A no exista ningún bi-ideal minimal (como ocurre en el semianillo de los enteros no negativos N), o que exista, al menos, uno. En este último caso A posee un único bi-ideal minimal, que es, por tanto, mínimo, el bi-ideal nuclear, por ser éste la intersección no vacía de todos los bi-ideales no triviales de A.

Por otra parte, si $A \neq 0$ contiene elemento distinguido w, A posee tambiém un único bi-ideal minimal no trivial, que, por tanto, es mínimo no trivial, precisamente $\mathbf{k}(A) = W$. Introducimos en este caso el concepto de w-minimalidad de bi-ideales.

Se dice que el bi-ideal $\mathbf{m}$ del w-semianillo A es w-minimal si y sólo si $W = \{w\}$ es el único bi-ideal no trivial de A contenido estrictamente en $\mathbf{m}$. Es evidente que dos bi-ideales w-minimales de A diferentes entre sí son w-disjuntos (su intersección es W).

Proposición 5

Todo bi-ideal w-*minimal* $\mathbf{m}$ *de un* w-*semianillo* A *que no sea* w-*simétrico es* R-*simple.*

En efecto, sea $\mathbf{m}$ bi-ideal w-minimal de A tal que

$$\mathbf{m} + \mathbf{m} \neq W, \quad \mathbf{m} \cdot \mathbf{m} \neq W,$$

mostraremos por reducción al absurdo que entonces $\mathbf{m}$ es R-simple.

Como $(\mathbf{m} + \mathbf{m})$ es bi-ideal de A contenido en $\mathbf{m}$ (proposición 1 iii)) y $\mathbf{m}$ es w-minimal o bien $\mathbf{m} + \mathbf{m} = W$, o bien $\mathbf{m} + \mathbf{m} = \mathbf{m}$. Lo primero no puede ser por hipótesis, luego

18) $\mathbf{m} + \mathbf{m} = \mathbf{m}$.

Sea $\mathbf{u}$ un bi-ideal de $\mathbf{m}$ comprendido estrictamente entre W y $\mathbf{m}$ y sea $u^* \in \mathbf{u}$. Entonces

$$\mathbf{t} = (\mathbf{m}\, u^*\, \mathbf{m}) + \mathbf{m}$$

es un bi-ideal de A contenido en $\mathbf{u}$; por la w-minimalidad de $\mathbf{m}$ resulta

19) $(\mathbf{m}\, u^*\, \mathbf{m}) + \mathbf{m} = W, \quad \forall\, u^* \in \mathbf{u}$.

Teniendo en cuenta 18) se obtiene:

$$\mathbf{m}\, u^*\, \mathbf{m} = \mathbf{m}\, u^* (\mathbf{m} + \mathbf{m}) \subseteq \mathbf{m}\, u^*\, \mathbf{m} + \mathbf{m}\, u^*\, \mathbf{m}.$$

Como

$$\mathbf{m}\, u^*\, \mathbf{m} \subseteq \mathbf{m}\,\mathbf{m}\,\mathbf{m} \subseteq A'\,\mathbf{m}\,\mathbf{m} \subseteq \mathbf{m}\,\mathbf{m} \subseteq A'\,\mathbf{m} \subseteq \mathbf{m},$$

se verifica, según 19),

$$\mathbf{m}\, u^*\, \mathbf{m} \subseteq \mathbf{m}\, u^*\, \mathbf{m} + \mathbf{m} = W, \quad \forall\, u^* \in \mathbf{u}.$$

Así pues,

20) $\mathbf{m}\, u^*\, \mathbf{m} = W, \quad \forall\, u^* \in \mathbf{u}$.

El conjunto

$$V(\mathbf{m}) = \{x \in \mathbf{m} / \mathbf{m}\, x\, \mathbf{m} = W\}$$

es un bi-ideal de A contenido en $\mathbf{m}$ y que contiene estrictamente a W, en virtud de 20). En virtud de lo antedicho y de la w-minimalidad de $\mathbf{m}$, se obtiene

$$V(\mathbf{m}) = \mathbf{m},$$

es decir,

21) $\mathbf{m}\,\mathbf{m}\,\mathbf{m} = W$,

lo que implica

$$\mathbf{m}\,\mathbf{m} \subset \mathbf{m} \quad \text{(estrictamente)}.$$

Como, por hipótesis, $\mathsf{m}\cdot\mathsf{m}\neq W$, resulta que $\mathsf{m}\cdot\mathsf{m}$ está estrictamente comprendido entre W y m:

$$W\subset\mathsf{m}\cdot\mathsf{m}\subset\mathsf{m}.$$

Atendiendo a $18)_2$ se verifica:

$$W\subset\mathsf{m}\cdot\mathsf{m}=\mathsf{m}\cdot(\mathsf{m}+\mathsf{m})\subseteq\mathsf{m}\cdot\mathsf{m}+\mathsf{m}\cdot\mathsf{m}\subseteq\mathsf{m}\cdot\mathsf{m}+\mathsf{m}.$$

Pero $(\mathsf{m}\cdot\mathsf{m}+\mathsf{m})$ es un bi-ideal de A que está contenido en m y que, según lo anterior, contiene estrictamente a W; por tanto,

$$\mathsf{m}\cdot\mathsf{m}+\mathsf{m}=\mathsf{m}.$$

Teniendo en cuenta esta igualdad junto con la 21), resulta

$$\mathsf{m}\cdot\mathsf{m}=(\mathsf{m}\cdot\mathsf{m}+\mathsf{m})\cdot\mathsf{m}\subseteq\mathsf{m}\cdot\mathsf{m}\cdot\mathsf{m}+\mathsf{m}\cdot\mathsf{m}\subseteq W+\mathsf{m}=W.$$

Así pues, llegamos a $\mathsf{m}\cdot\mathsf{m}=W$, lo que es una contradicción, c. q. d.

En otro artículo [8] mostraremos con un ejemplo que si m es bi-ideal w-minimal de A y se cumple

$$\mathsf{m}+\mathsf{m}=\mathsf{m}, \qquad \mathsf{m}\cdot\mathsf{m}=W,$$

entonces hay casos en los que existen bi-ideales propios de m. Por lo tanto, si en lugar de definir semianillo w-simétrico como lo hemos hecho, lo hubiásemos definido de esta manera: «A es semianillo w-simétrico si y sólo si es un w-semianillo en el que

$$A'\neq W, \qquad A'+A'=W\text{»},$$

y hubiéramos dejado invariable la definición de semianillo R-simple, no se verificaría la proposición 5.

Sea T subsemianillo del semianillo A; designará $T_0=T\cup\{0\}$, caso de que A contenga un cero «0» que no está en T, $T_0=T$, caso contrario. T_0 constituye también un subsemianillo de A.

Nota a la proposición 5: La proposición 5 es válida tanto si A contiene un cero como si no lo contiene. En dicha proposición se verifican las equivalencias «m es w-simétrico $\Longleftrightarrow$ m_0 es w-simétrico», «m es R-simple $\Longleftrightarrow$ m_0 es R-simple», debido, entre otras razones, a que m no puede contener un cero «u» distinto del cero «0» de A,

ya que $w\in\mathsf{m}$, $w\cdot u=w$. Por esto mismo, m, en tanto semianillo, verifica $\mathsf{m}'=\mathsf{m}$.

Proposición 6

$\mathsf{k}(\mathrm{A})_0=\mathsf{k}(\mathrm{A})\cup\{0\}$ *constituye un semianillo* R-*simple, siendo* $\mathsf{k}(\mathrm{A})$ *el bi-ideal nuclear del semianillo con cero* A.

En efecto, si A es w-semianillo, $\mathsf{k}(A)_0=W_0=O_w$, que ya hemos visto que es R-simple.

Supongamos, pues, que A no contiene elemento distinguido y que u es un bi-ideal propio de $\mathsf{k}(A)_0$, entonces

$$\mathsf{l}=\mathsf{k}(A)\cdot\mathsf{u}\cdot\mathsf{k}(A)+\mathsf{k}(A)$$

sería un bi-ideal no vacío de A contenido en u, el cual está contenido estrictamente en $\mathsf{k}(A)$, contradicción, c. q. d.

Nota: No es verdad, en general, que $\mathsf{k}(A)$ sea semianillo R-simple debido a que $\mathsf{k}(A)$ puede contener un cero $u\neq 0$ diferente del cero de A. En un ejemplo que pondremos en [8] se manifiesta que $\mathsf{k}(A)$ no es R-simple.

En virtud de la proposición 1 iii) y de las definiciones, se verifica

Proposición 7

Sea A *un* w-*semianillo que sólo contiene como bi-ideales no triviales a* A′ *y a* W, *entonces* A *es* w-*simétrico primitivo ó es* R-*simple.*

BIBLIOGRAFIA

[1] Clifford, A. H. and Preston, G. B., *The Algebraic Theory of Semigroups.* Am. Mat. Soc. 1961.

[2] Gluskin, L. M., *Bi-ideales de semigrupos* (en ruso) Izv. Vysš. Učebn Zaved Matematika 1970 n.° 9 **(100)** 29-36.

[3] Good, R. A. and Hughes, D. R., *Associated groups for a semigroup.* Bull. Amer. Math. Soc. **58** (1952) 624-625. (Abstract).

[4] Green, J. A., *On the structure of semigroups.* Annals of Math. **54** (1951) 163-172.

[5] LAJOS, S.; SZÁSZ, FERENC, A., *On the bi-ideals in associative rings.* Proceed. Japan Acad. **46** (1970) 505-507.

[6] POYATOS, F., *Introducción a la teoria de semimódulos* (tesis doctoral). Publicaciones de la Facultad de Ciencias, Madrid, 1967.

[7] POYATOS, F., *El teorema de Jordan-Hölder para A-semimódulos.* Rev. Mat. Hisp.-Am. 4.ª Serie, tomo XXXII, núm 6 (1972) 251-260, tomo XXXIII núm 1-2 (1973) 36-48, núm 3 (1973) 122-132.

[8] POYATOS, F., *Sobre ciertas propiedades de los bi-ideales de un semianillo.* Aparecerá en Revista da Faculdade de Ciências de Lisboa.

[9] REES, D., *On semi-groups.* Proc. Cambridge Phil. Soc. **36** (1940) 387-400.

[10] SZÁSZ, FARENC, A., *Generalized bi-ideals of rings*, I, II, Math. Nach. **47** (1970) 355-360 ibid **47** (1970) 361-364.

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ESPAÑA

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
Lisboa, 1974

ON SUPERNILPOTENT RADICALS (*)

BY

S. A. AMITSUR

To Professor Dr. ALMEIDA COSTA

1. INTRODUCTION

Let K be a class of associative rings closed under homomorphisms and taking right and left ideals. A subclass $N \subseteq K$ is a *radical* class if it satisfies ([1], [3]):

(RA) Every ring $R \in K$ contains a maximal ideal $N(R) \in N$ and $R/N(R)$ has no non-zero ideals which belong to N.

Most known radicals have additional properties as:

(RB) The right and left ideals in $N(R)$ belong to N (*Hereditary* radicals).

(RC) $N(R)$ contains all one sided ideals of R wich belong to N (*Strong* radicals).

A ring R is a zero ring if $R^2 = 0$, i. e. the multiplication in R is trivial: $xy = 0$ for $x, y \in R$.

A radical class is *supernilpotent* in K if N contains all zero rings, and hence all nilpotent ideals.

BAER's Lower Radical is the minimal radical which contains all zero rings, and hence all supernilpotent radical contain the Lower Radical. DIVINSKY has given in [3] examples of radical classes which are smaller than the Lower Radical. The examples of DIVINSKY depends

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heavily on the additive structure of the rings involved. In the summary [2], we have pointed out that this is to be expected since the strong hereditary radicals can be obtained in two steps; the first depends only on the additive structure, i. e. on torsion theory for abelian groups and the second step is taking a supernilpotent radical.

In the present note we provide the proofs and a more detailed discussion of these radicals. In fact, we require a little less than a strong hereditary radicals.

(*) Henceforth, N will be a radical class satisfying in addition to (RA) also:

(RB_1) If $R \in N$ and A is an ideal in R such that $A^2=0$ then A also belongs to N.

(RC_1) The sum of a finite number left N-ideals (i. e. ideals $\in N$) is a left N-ideal.

Note that (RB_1) is a little less than hereditary (RB) and (RC_1) is less strict than the strongness (RC). As strong radicals satisfy the result that sum of left N-ideals is a left N-ideal.

2. RADICALS OF THE TYPE (*)

Let $R \in K$, we denote by R^+ the additive group of elements of R turned into a zero ring by the trivial multiplication, i. e. $xy=0$ for $x, y \in R$. We begin with a basic result of Krempa [4]:

Lemma 1: *If* N *satisfies* (*) *then for* $\mathrm{R} \in \mathrm{N}$, *also* $\mathrm{R}^+ \in \mathrm{N}$.

Proof: Consider the ring $R_2 = \{(a, b);\ a, b \in R\}$ with addition defined componentvise and multiplication given by: $(a, b)(x, y) = (ax, ay)$. R_2 is a ring isomorphic with one row matrices in the 2×2 matrix ring over R. This ring contains two left ideals $L_1 = \{(a, 0),\ a \in R\}$, $L_2 = \{(a, a),\ a \in R\}$ which are isomorphic to R by the maps: $a \to (a, 0)$ for L_1, and $a \to (a, a)$ for L_2. Hence, $R \in N$ implies that $L_1, L_2 \in N$. Furthermore, $L_1 + L_2 = R_2$ which in view of (RC_1) it follows that $R_2 \in N$. Now, R_2 contains a two sided ideal $L_3 = \{(0, a),\ a \in R\}$ and $L_3^2 = 0$. Thus, (RB_1) yields that $L_3 \in N$. Finally, $L_3 \cong R^+$ by mapping: $(0, a) \to a$ hence $R^+ \in N$.

Q. E. D.

With the aid of the class N we define a larger radical class in K, which we denote by K_N:

Definition: $R \in K_N$ if the zero ring $R^+ \in N$.

Our first result is:

Theorem 2: *If* N *is of type* (*) *then* $\mathrm{K_N}$ *is a strong hereditary radical class in* K.

Proof: One of the many methods of proving that a class is radical is to show its closure under homomorphisms, extensions and union of non-decreasing sequence of ideals of the class [1]. In our case the proof is straightforward:

Let $R \in K_N$, then $R^+ \in N$. Also, for any homomorphism φ of R, $(R\varphi)^+ = R^+\varphi$ and so $(R\varphi)^+ \in N$ which yield $R\varphi \in K_N$ (closure under homomorphisms).

If P is an ideal in R and $P \in K_N$, $R/P \in K_N$. Then P^+ and $(R/P)^+ \in N$. Clearly $(R/P)^+ \cong R^+/P^+$, hence $R^+ \in N$ since N is closed under extension. Thus, $R \in K_N$ (Closure under extensions).

Let $\{L_\alpha\}$ be a non-decreasing transfinite sequence of ideals in R. If all $L_\alpha \in K_N$, then $L_\alpha^+ \in N$. Each L_α^+ is an ideal in R^+ then the fact $(\bigcup L_\alpha)^+ = \bigcup L_\alpha^+$ and that N is closed under union of such sequence imply that $(\bigcup L_\alpha)^+ \in N$ and so $\bigcup L_\alpha \in K_N$ (Closure under unions).

This completes the proof that K_N is a radical class. K_N is even strong and hereditary: For let $R \in K_N$ and A a one-sided ideal in R, by definition $R^+ \in N$ and thus A^+ is a two-sided (!) ideal and $A^{+2}=0$. Hence (RB_1) implies that $A^+ \in N$ and consequently $A \in K_N$, which proves that K_N is hereditary.

To show that K_N is a strong radical class, let A be a left ideal in a ring R and $A \in K_N$ then $A^+ \in N$ and A^+ is a zero ring in R^+. For every $x \in R$ the correspondence x_r: $a \to ax$, $a \in A$ defines a ring homomorphism of A^+ onto $A^+x = (Ax)^+$ as both are zero rings, hence $(Ax)^+ \in N$. Now, N is a radical class and therefore sum of N-ideals is an N-ideal and this yields that $\sum_{x \in R} (Ax)^+ + A^+ = (A + AR)^+$ is an N-ideal in R^+. Thus $(A + AR)^+ \in N$ and hence

$A + AR \in K_N$. Consequently, every left, and similarly right, ideal in K_N is contained in a two sided ideal in K_N. This property is known to be equivalent to (RC), an this completes the proof that K_N is strong.

Since K_N is a radical class, every ring will contain a radical $K_N(R)$, and our next result is:

THEOREM 3: (i) $K_N \supseteq N$ *and* N *defines a supernilpotent radical in the class* K_N. ii) *For every ring* $R \in K$, $N(R) = N[K_N(R)]$.

PROOF: Lemma 1 is the statement that $K_N \supseteq N$. N is a supernilpotent radical in K_N, for let R be a zero ring in K_N, i. e. $R^+ \in N$. But for zero rings $R = R^+$ and thus $R \in N$, i. e. N is supernilpotent.

To complete the proof of (i) we have still to show that K_N is an admissible class, i. e., it is closed under homomorphism and taking left ideals: Indeed, K_N closed under homomorphism as shown in theorem 2 (for K_N is a radical class in K); and if L is a left ideal in R, $R \in K_N$ then $R^+ \in N$ and then L^+ is a zero (two sided) ideal in R^+. It follows, therefore, by (RB_1) that $L^+ \in N$ which means that $L \in K_N$, which completes the proof of (i).

To prove (ii) we first note that since $N(R) \in N$ then by lemma 1, $N(R)^+ \in N$, i. e. $N(R) \in K_N$. Consequently, $N(R) \subseteq K_N(R)$. $N[K_N(R)]$ is the maximal N-ideal in $K_N(R)$ hence, $N(R) \subseteq N[K_N(R)]$. To prove the converse, we note that for any radical class, e. g. N, the radical $N(A)$ of an ideal A in a ring R is an ideal in R. Hence in our case $N[K_N(R)]$ is an ideal in R and belongs to N, consequently $N[K_N(R)] \subseteq N(R)$.

3. TORSION THEORY DETERMINED BY K_N.

The preceding results show that we obtain $N(R)$ in two steps: first, taking the radical $K_N(R)$ and then a supernilpotent radical of $K_N(R)$, namely $N[K_N(R)]$.

Our next aim is to show $K_N(R)$ is actually a torsion subgroup of the additive group of R with respect to a certain torsion theory. This will be shown for classes K which contain all zero rings. The method can be applied also for a more general case.

We assume, henceforth, that K contains the class K_0 of all zero rings. This class can be identified with the category of all abelian groups — by turning every abelian group A into a zero ring through the trivial multiplication: $ab = 0$ for $a, b \in A$.

The correspondence $R \to R^+$ defines the forgetful functor from K into K_0 which corresponds to each R the additive group of the elements of R. It is the identity on K_0 and furthermore, lemma 1 implies that this functor maps N into N_0, where $N_0 = N \cap K$.

THEOREM 4: N_0 *is a torsion class in* K_0. *For every ring* $R \in K$, $K_N(R)$ *is the torsion subgroup of the additive group of* R *with respect to this torsion theory.*

PROOF: A subclasse of the category of all abelian groups is a torsion class and determines a torsion theory ([5]) if it is closed under injection, surjection, extensions and direct limits. The proof that N_0 satisfies these conditions is almost an immediate consequence from lemma 1 and the properties of the class N. We prove here only the closure under injections and extensions and proof of the rest is similar:

Closure under injections: Let $A \in N_0$ and B a subgroup of A, then B is also an ideal in A and $B^2 = 0$, hence by (RB_1) it follows that also $B \in N_0$.

Closure under extensions: Let $0 \to A \to B \to C \to 0$ be an exact sequence in K_0 and such that $A, C \in N_0$. As rings $A, C \in N$ and so $B \in N$, since N is closed under extensions. Hence $B \in N \cap K_0 = N_0$.

For any ring R, let $N_0(R^+)$ be the maximal N_0-torsion subgroup of R^+, i. e. of the additive subgroup of R. Our theorem theorem states that $N_0(R^+) = K_N(R)^+$.

Indeed, by definition $K_N(R)$ is the maximal ideal of R such that $K_N(R)^+ \in N$; but $K_N(R)^+$ is a zero ring, hence $K_N(R)^+ \in N_0$. Consequently, $N_0(R^+) \supseteq K_N(R)^+$. To prove the converse, we first show that $N_0(R^+)$ is actually an ideal in R. For, let $x \in R$ then the map: $N_0(R^+) \to N_0(R^+)x$ defined by corresponding $r \to rx$ is a homomorphism of zero additive groups (i. e. zero rings) and $N_0(R^+) \in N_0$ thus also $N_0(R^+)x \in N_0$. But $N_0(R^+)$ is the maximal N_0-subgroup of R^+, hence $N_0(R^+)x \subseteq N_0(R^+)$. Similarly, one shows that $xN_0(R^+) \subseteq N_0(R^+)$ for every $x \in R$. Both these relations imply that $N_0(R^+)$ is a two sided ideal in R. The additive group of the ideal $N_0(R^+)$ belongs to N_0 by definition of $N_0(R^+)$, and this means that $N_0(R^+)$ as an ideal belongs to K_N. Consequently $N_0(R^+) \subseteq K_N(R)^+$.

REMARK 1: The last result means that in order to obtain the radical $N(R)$, we first consider the maximal N_0-torsion subgroup, i.e. $K_N(R)$ and then determine the radical $N[K_N(R)]$ in the class of all rings of K whose additive group is N_0-torsion, and the latter is a supernilpotent radical.

REMARK 2: If K is a class of all Ω-algebras, where Ω is a commutative ring. If we assume that N is a radical of Ω-algebras and $K_0 \subseteq K$, where K_0 is the category of all Ω-modules turned into zero rings. Then one can extend the preceding results in an obvious way and show that $N_0 = N \cap K_0$ is a torsion theory in the category of all Ω-modules and all the other results.

BIBLIOGRAPHY

[1] S. A. AMITSUR, *A general theory of radicals II*, Am. J. Math. **76** (1954), pp. 100-125.
[2] S. A. AMITSUR, *Nilradicals: Historical notes and new results*. Colloq. Math. Soc. Janos Bolyai de Rings, Modules and Radicals, Keszthely (Hungary), 1971, pp. 47-65.
[3] H. J. DIVINSKY, *Rings and radicals*, Univ. of Toronto Press.
[4] J. KREMPA, *Radical of semigroup rings*, Fund. Math. (to appear).
[5] BO STENSTROM, *Rings and modules of quotients*, Springer Verlag, 1971, Ch. I.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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DEMI-ANNEAUX: MORPHISMES ET IDÉAUX II (*)

PAR

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Hommage simple à Mr. le Prof. ALMEIDA COSTA

1) Introduction — Le concept de demi-anneau $\mathfrak{S}$ ainsi que les concepts d'idéal $\mathfrak{a}$ et d'idéal nucléaire $\mathfrak{J}$, de $\mathfrak{S}$, ont été donnés dans la partie I de ce travail [7]. Et il en est de même en ce qui concerne les notions d'épimorphisme, de semi-isomorphisme, de noyau d'un épimorphisme, etc. Rappelons, toutefois, qu'un idéal contenant un seul élément será désigné par idéal singulier. D'une manière générale, le lecteur à qui puissent apparaître de doutes dans l'emploi de quelques mots dont on fasse usage, on recommande l'ouvrage d'ALMEIDA COSTA [2] et, naturellement, le travail [7].

Étudier les congruences sur un système algèbrique équivaut, comme on le sait, à étudier les images épimorphes du système. Dans le cas des anneaux tout idéal définit une congruence et toute congruence est définie par un idéal. Dans la partie I de ce travail, nous avons vu que la situation pour les demi-anneaux est tout à fait différente. Nous ferons à ce propos quelques

REMARQUES: I) Soit $\mathfrak{S}/\rho$ un demi-anneau-quotient. Il existe au plus une classe contenant un idéal, et ce fait est en effet réalisé, si et seulement si $\mathfrak{S}/\rho$ contient un élément qui soit à la fois idempotent additif et absorbant multiplicatif.

II) Les classes ayant des éléments communs avec un idéal $\mathfrak{a}$, de $\mathfrak{S}$, constituent un idéal de $\mathfrak{S}/\rho$, lequel est l'image de $\mathfrak{a}$ dans

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l'épimorphisme canonique $\mathfrak{S} \to \mathfrak{S}/\rho$; et l'ensemble union des classes en question est de même un idéal de $\mathfrak{S}$, d'ailleurs saturé pour la relation de congruence ρ et ayant la même image que $\mathfrak{a}$.

III) Les classes ayant des éléments communs avec $\mathfrak{a}$ sont des identités à droite (à gauche) additives de $\mathfrak{S}/\rho$, si et seulement si $x + \mathfrak{a} \subseteq C_x$, $\forall x \in \mathfrak{S}$, $(\mathfrak{a} + x \subseteq C_x$, $\forall x \in \mathfrak{S})$. En particulier, si les éléments de $\mathfrak{S}$ appartenant à une classe de $\mathfrak{S}/\rho$ constituent un idéal $\mathfrak{a}$, de $\mathfrak{S}$, et la classe est une identité à droite (à gauche) de $\mathfrak{S}/\rho$, on a: 1) $x + a \in \mathfrak{a}$, $a \in \mathfrak{a} \Rightarrow x \in \mathfrak{a}$; 1′) $a + x \in \mathfrak{a}$, $a \in \mathfrak{a} \Rightarrow x \in \mathfrak{a}$.

Un idéal $\mathfrak{a}$, d'un demi-anneau $\mathfrak{S}$, vérifiant cette condition 1), est dit un k_d-idéal [et si vérifie 1′) est appelé un k_e-idéal]. Lorsque $\mathfrak{a}$ vérifie à la fois 1) et 1′), $\mathfrak{a}$ est un k-idéal [5].

Voici deux propositions faisant intervenir ces notions:

Proposition 1: *Étant donné un morphisme* $\mathfrak{S} \to \mathfrak{S}'$ *de demi-anneaux, l'image complète inverse d'un k-idéal (d'un k_d-idéal; d'un k_e-idéal) de* $\mathfrak{S}'$ *est un k-idéal (un k_d-idéal; un k_e-idéal) de* $\mathfrak{S}$.

Proposition 2: *Étant donné un épimorphisme* $\mathfrak{S} \to \mathfrak{S}'$ *de demi-anneaux, si l'image complète inverse d'un sous-ensemble de* $\mathfrak{S}'$ *est un k-idéal (k_d-idéal, k_e-idéal) de* $\mathfrak{S}$, *alors le sous-ensemble est un k-idéal (k_d-idéal, k_e-idéal de* $\mathfrak{S}'$.

D'une façon succincte, il s'agit dans ce travail d'approfondir des résultats obtenus dans [7]. Nous étendons effectivement le concept de B-idéal et démontrons des théorèmes correspondants, qui sont ainsi des extensions des théorèmes donnés dans [7]. De même, nous revenons aux Q-idéaux, que nous étendons aussi et pour lesquels nos nouveaux résultats sont des extensions des anciens.

2. Congruences de Bourne. B-idéaux — Soit β une relation de Bourne définie par un idéal $\mathfrak{a}$ [3], [7]: $x \beta y \Longleftrightarrow \exists\, a_1, a_2 \in \mathfrak{a}$, tels que $x + a_1 = y + a_2$. Cette relation est réflexive et symétrique et vérifie les relations

i) $(x + a) \beta x$, $\forall x \in \mathfrak{S}$;

ii) $(x + a_1) \beta (y + a_2)$, $a_1, a_2 \in \mathfrak{a} \Rightarrow x \beta y \Rightarrow (z + x) \beta (z + y)$, $\forall z \in \mathfrak{S}$;

mais elle n'est pas en général une relation de congruence.

Nous représenterons par $\beta_{\mathfrak{a}}$ la relation de Bourne définie par $\mathfrak{a}$. Les deux propositions à suivre sont valables:

Proposition 1: $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \Rightarrow (\beta_{\mathfrak{a}_1} \Rightarrow \beta_{\mathfrak{a}_2})$.

Proposition 2: $\beta_{\mathfrak{a}}$ *implique toute relation de congruence* ρ *sur* $\mathfrak{S}$ *dont les classes ayant des éléments communs à* $\mathfrak{a}$ *soient des identités à droite additives de* $\mathfrak{S}/\rho$. En effet: $x \beta y$ implique $x + a_1 = y + a_2$, avec $a_1, a_2 \in \mathfrak{a}$. En allant aux classes correspondantes de $\mathfrak{S}/\rho$, on obtient $D_x + D_{a_1} = D_y + D_{a_2}$, et, puisque D_{a_1} et D_{a_2} sont des identités à droite additives, on en conclut $D_x = D_y$, donc $x \rho y$.

D'une façon apparenté à [7], si $\mathfrak{a}$ définit une relation de Bourne de congruence on le dit un *B-idéal* ou un idéal de Bourne et le demi-anneau-quotient $\mathfrak{S}/_{\beta}\mathfrak{a}$ correspondant est appelé demi-anneau-quotient de Bourne.

D'après i), toute classe C_a, $(a \in \mathfrak{a})$, de $\mathfrak{S}/_{\beta}\mathfrak{a}$, est une identité à droite additive.

Remarques: I) Si $\mathfrak{S}$ est un G_e-demi-anneau [demi-anneau dont les éléments sont des identités à gauche additives], alors tous les idéaux non vides sont des B-idéaux définissant la congruence universelle.

II) Tout idéal dont les éléments soient des identités à droite additives est un B-idéal définissant la congruence identité. En particulier cela arrive pour tout idéal non vide d'un G_d-demi-anneau $[x + y = y$, $\forall x, y \in G_d]$.

Théorème 1: *Si* $\mathfrak{a}$ *est un B-idéal d'un demi-anneau* $\mathfrak{S}$, *il en est de même de tout idéal* $\mathfrak{a}'$ *tel que* $\mathfrak{a} \subseteq \mathfrak{a}' \subseteq \bigcup_{a \in \mathfrak{a}} C_a$, *et on a*

$$\mathfrak{S}/_{\beta}\mathfrak{a} = \mathfrak{S}/_{\beta}\mathfrak{a}' = \mathfrak{S}/_{\beta} \bigcup_{a \in \mathfrak{a}} C_a .$$

D'après la proposition 1, $\beta_{\mathfrak{a}} \Rightarrow \beta_{\mathfrak{a}'} \Rightarrow \beta_{\bigcup_{a \in \mathfrak{a}} C_a}$, donc, compte tenu de la proposition 2, $\beta_{\bigcup_{a \in \mathfrak{a}} C_a} \Rightarrow \beta_{\mathfrak{a}}$, ce qui entraîne $\beta_{\mathfrak{a}} = \beta_{\mathfrak{a}'} = \beta_{\bigcup_{a \in \mathfrak{a}} C_a}$.

Théorème 2: $\bigcup_{a \in \mathfrak{a}} C_a$ *est le k_d-idéal engendré par* $\mathfrak{a}$. Puisque tout C_a, $(a \in \mathfrak{a})$, est une identité à droite additive de $\mathfrak{S}/_{\beta}\mathfrak{a}$, l'ensemble $\{C_a \mid C_a \in \mathfrak{S}/_{\beta}\mathfrak{a},\ a \in \mathfrak{a}\}$ est un k_d-idéal de $\mathfrak{S}/_{\beta}\mathfrak{a}$. La proposition 1, n.º 1, montre que $\bigcup_{a \in \mathfrak{a}} C_a$ est un k_d-idéal de $\mathfrak{S}$. Alors, soit $\mathfrak{a}_0$ un k_d-idéal contenant $\mathfrak{a}$. Si $x \in \bigcup C_a$, on a $x \in C_a$, pour un certain a, et $x + a_1 = a + a_2$, pour certains $a_1, a_2 \in \mathfrak{a}$. Compte tenu de $a_1, a + a_2 \in \mathfrak{a}_0$, on tire $x \in \mathfrak{a}_0$ et $C_a \subseteq \mathfrak{a}_0$, ce qui prouve l'inclusion $\bigcup C_a \subseteq \mathfrak{a}_0$.

Corollaire 1: *Un B-idéal $\mathfrak{a}$, de $\mathfrak{S}$, est un k_d-idéal, si et seulement si* $\mathfrak{a} = \bigcup_{a \in \mathfrak{a}} C_a$. D'ailleurs, un k_d-idéal qui contienne un B-idéal est saturé pour la relation de Bourne définie par celui-ci.

Théorème 3: *La relation de congruence définie par un B-idéal $\mathfrak{a}$, de $\mathfrak{S}$, entraîne la relation de congruence ρ sur $\mathfrak{S}$, si et seulement si toute classe appartenant à $\mathfrak{S}/\rho$ qui ait des éléments communs avec $\mathfrak{a}$ soit une identité à droite additive.* Si $\beta \Rightarrow \rho$, on a un épimorphisme canonique $\mathfrak{S}/_{\beta}\mathfrak{a} \to \mathfrak{S}/\rho$, et par conséquent toute classe de $\mathfrak{S}/\rho$ qui ait des éléments communs avec $\mathfrak{a}$ est une identité à droite additive, car elle est une image d'une classe de $\mathfrak{S}/_{\beta}\mathfrak{a}$ dans les mêmes conditions. La réciproque est une conséquence de la proposition 2.

Proposition 3: *En supposant $\mathfrak{a}$ un B-idéal, les trois assertions suivantes sont équivalentes:*

i) $\mathfrak{S}/_{\beta}\mathfrak{a}$ *a l'élément-zéro.*

ii) *Étant donné $x \in \mathfrak{S}$, il existe $a_1, a_2, a_3 \in \mathfrak{a}$ tels que* $a_1 + x + a_2 = x + a_3$.

iii) *Etant donné $x \in \mathfrak{S}$ et $a \in \mathfrak{a}$, il existe $a_2, a_3 \in \mathfrak{a}$ tels que* $a + x + a_2 = x + a_3$.

On dit $\mathfrak{a}$ un B_0-idéal de $\mathfrak{S}$, lorsque $\mathfrak{a}$ est un B-idéal vérifiant l'une quelconque des conditions de la proposition 3. Évidemment que, en supposant $\mathfrak{a}$ un B_0-idéal, $\mathfrak{a}$ est contenu dans la classe qui est l'élément-zéro de $\mathfrak{S}/_{\beta}\mathfrak{a}$. (0), s'il existe, est un B_0-idéal.

Remarques: I) Dans un demi-anneau G_e, tous les idéaux non vides sont B_0-idéaux.

II) Dans un demi-anneau G_d, ayant plus d'un élément, il n'existe aucun B_0-idéal.

Théorème 4: *Soit $\mathfrak{a}$ un B_0-idéal de $\mathfrak{S}$. Alors $C_a \in \mathfrak{S}/_{\beta}\mathfrak{a}$, $(\mathfrak{a} \subseteq C_a)$, est le k-idéal engendré par $\mathfrak{a}$.*

Corollaire 2: *Un B_0-idéal $\mathfrak{a}$ est un k-idéal, si et seulement si $C_a = \mathfrak{a}$.*

En tant qu'une conséquence du théorème 3, on a:

Proposition 4: *Un B-idéal qui contienne un B_0-idéal est lui même un B_0-idéal.* En conséquence, si l'on admet que l'idéal nucléaire est un B_0-idéal, tout B-idéal est un B_0-idéal.

Conformément à [7], $\mathfrak{a}$ est dit normal, lorsque $\forall x \in \mathfrak{S}$, on a $\mathfrak{a} + x \subseteq x + \mathfrak{a}$.

Remarques: I) L'idéal vide est normal.

II) Un demi-anneau contient un idéal normal singulier $\{z\}$, si et seulement si z est un elément idempotent additif et absorbant multiplicatif appartenant au centre du demi-groupe additif du demi-anneau.

III) Si $0 \in \mathfrak{S}$, alors (0) est un idéal normal.

Proposition 5: *Si $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ est un épimorphisme de demi-anneaux, l'image d'un idéal normal de $\mathfrak{S}$ est un idéal normal de $\mathfrak{S}'$.*

Théorème 4: *Un idéal normal $\mathfrak{a} \neq \emptyset$ est un B_0-idéal.* Nous savons qu'il définit une relation de congruence [7], donc il s'agit d'un B-idéal. Soit maintenant C_a, $(a \in \mathfrak{a})$, une classe appartenant à $\mathfrak{S}/_{\beta}\mathfrak{a}$. Alors C_a est une identité à droite additive, et puisque $\mathfrak{a} + x \subseteq x + \mathfrak{a} \subseteq C_x$, on voit aussi que toute classe ayant des éléments communs avec $\mathfrak{a}$ est une identité à gauche additive. En conséquence, il n'existe qu'une identité additive dans $\mathfrak{S}/_{\beta}\mathfrak{a}$ qui est la classe contenant $\mathfrak{a}$ et qui est l'élément-zéro du quotient.

Corollaire 3: *Dans un demi-anneau commutatif, tout idéal est un B_0-idéal.*

Remarques : I) Un G_e-demi-anneau n'a que deux idéaux normaux : $\emptyset$ et $\mathfrak{S}$.

II) Un G_d-demi-anneau ayant plus d'un élément n'a que l'idéal vide en tant que normal.

III) Un B_0-idéal n'est pas nécessairement un idéal normal [Remarques I') et I'')].

3) Un théorème d'isomorphisme par rapport aux B-idéaux — LaTorre [5] établit pour les demi-anneaux commutatifs ayant l'élément-zéro un théorème analogue au théorème d'isomorphisme de la Théorie des Anneaux, mais en imposant aux demi-anneaux qu'il soient du type (H) ou du type (K). Dans ce qui suit, nous donnons un théorème d'isomorphisme pour n'importe quels demi-anneaux, de telle sorte que son application aux demi-anneaux commutatifs (avec ou sans élément-zéro) est indépendant du fait qu'il s'agit de demi-anneaux des types (H) ou (K).

Théorème 1 : *Soient* $\mathfrak{S}\xrightarrow{\varphi}\mathfrak{S}'$ *un épimorphisme de demi-anneaux et* $\mathfrak{a}$ *un idéal de* $\mathfrak{S}$ *tel que* $\varphi(x)=\varphi(y)\Rightarrow x\beta_{\mathfrak{a}}y$; *alors* $\mathfrak{a}$ *est un B-idéal, si et seulement si* $\varphi(\mathfrak{a})$ *est un B-idéal de* $\mathfrak{S}'$. *Si cela arrive, l'isomorphisme suivant est valable :*

$$\mathfrak{S}/_{\beta}\mathfrak{a}\simeq\mathfrak{S}'/_{\beta}\varphi(\mathfrak{a}).$$

Compte tenu de ce qu'on sait de la théorie générale des systèmes algébriques, il suffit de vérifier l'équivalence

$$x\beta_{\mathfrak{a}}y\Leftrightarrow\varphi(x)\beta_{\varphi(\mathfrak{a})}\varphi(y).$$

Or, $x\beta_{\mathfrak{a}}y$ implique $x+a_1=y+a_2$, pour certains $a_1,a_2\in\mathfrak{a}$; en conséquence, $\varphi(x)+\varphi(a_1)=\varphi(y)+\varphi(a_2)$, donc $\varphi(x)\beta_{\varphi(\mathfrak{a})}\varphi(y)$. Réciproquement, $\varphi(x)\beta_{\varphi(\mathfrak{a})}\varphi(y)$ implique $\varphi(x)+\varphi(a_1)=\varphi(y)+\varphi(a_2)$, avec $a_1,a_1\in\mathfrak{a}$, donc $\varphi(x+a_1)=\varphi(y+a_2)$ et $(x+a_1)\beta_{\mathfrak{a}}(y+a_2)$ ainsi que $x\beta_{\mathfrak{a}}y$.

Corollaire 1 : *Si* $\mathfrak{b}$ *est un B-idéal de* $\mathfrak{S}$ *et* $\mathfrak{a}$ *est un idéal du même demi-anneau tel que* $\mathfrak{b}\subseteq\mathfrak{a}$, *alors l'image* $\bar{\mathfrak{a}}$, *de* $\mathfrak{a}$, *par l'épimorphisme canonique* $\mathfrak{S}\to\mathfrak{S}/_{\beta}\mathfrak{b}$ *est un B-idéal de* $\mathfrak{S}/_{\beta}\mathfrak{b}$, *si et seulement si* $\mathfrak{a}$ *est un B-idéal de* $\mathfrak{S}$. *Dans ce cas l'isomorphisme suivant est valable :*

$$\mathfrak{S}/_{\beta}\mathfrak{a}\simeq(\mathfrak{S}/_{\beta}\mathfrak{b})/_{\beta}\bar{\mathfrak{a}}.$$

Théorème 2 : *Si* $\mathfrak{b}$ *est un B-idéal d'un demi-anneau* $\mathfrak{S}$ *et* $\mathfrak{a}\supseteq\mathfrak{b}$ *est un* k_d*-idéal (k-idéal), alors l'image* $\bar{\mathfrak{a}}$, *de* $\mathfrak{a}$ *par l'épimorphisme canonique* $\mathfrak{S}\to\mathfrak{S}/_{\beta}\mathfrak{b}$ *est un* k_d*-idéal (k-idéal) de* $\mathfrak{S}/_{\beta}\mathfrak{b}$, *et on peut écrire* $\bar{\mathfrak{a}}=\mathfrak{a}/_{\beta}\mathfrak{b}$. De $C_x+C_{a_1}=C_{a_2}$, avec $a_1,a_2\in\mathfrak{a}$, on tire $C_{x+a_1}=C_{a_2}$, c'est-à-dire $x+a_1+b_1=a_2+b_2$, avec $b_1,b_2\in\mathfrak{b}$. Puisque $\mathfrak{b}\subseteq\mathfrak{a}$ et $\mathfrak{a}$ est k_d-idéal, il en résulte $x\in\mathfrak{a}$ et $C_x\in\bar{\mathfrak{a}}$. Vérifions ensuite que $\mathfrak{a}$ est saturé pour la relation $\beta_{\mathfrak{b}}$. Soit $x\in\mathfrak{S}$ et $x\beta_{\mathfrak{b}}a$, $a\in\mathfrak{a}$. Alors $x+b_1=a+b_2$, avec $b_1,b_2\in\mathfrak{b}$, et les conditions $a+b_2\in\mathfrak{a}$, $b_1\in\mathfrak{a}$, entraînent $x\in\mathfrak{a}$, ce qui achève la demonstration.

Remarques : I) On remarquera que, en supposant $\mathfrak{b}$ un B-idéal tel que $\bar{\mathfrak{b}}=\{C_b\mid b\in\mathfrak{b},\ C_b\in\mathfrak{S}/_{\beta}\mathfrak{b}\}$ est l'idéal nucléaire de $\mathfrak{S}/_{\beta}\mathfrak{b}$ (comme cela arrive, par exemple, si $\mathfrak{b}$ a un seul élément), alors l'image complète inverse d'un idéal de $\mathfrak{S}/_{\beta}\mathfrak{b}$ est un idéal de $\mathfrak{S}$ qui contient $\mathfrak{b}$. Dans ces conditions, on peut ajouter :

II) L'image complète inverse $\mathfrak{a}$, d'un idéal $\bar{\mathfrak{a}}$, de $\mathfrak{S}/_{\beta}\mathfrak{b}$, par l'épimorphisme canonique $\mathfrak{S}\to\mathfrak{S}/_{\beta}\mathfrak{b}$, est un B-idéal de $\mathfrak{S}$, si et seulement si $\bar{\mathfrak{a}}$ en est un du quotient.

III) Les k_d-idéaux de $\mathfrak{S}/_{\beta}\mathfrak{b}$ sont les demi-anneaux $\mathfrak{a}/_{\beta}\mathfrak{b}$, où $\mathfrak{a}$ est un k-idéal de $\mathfrak{S}$ contenant $\mathfrak{b}$.

1er Théorème de l'isomorphisme : *Si* $\mathfrak{b}$ *et* $\mathfrak{a}$ *sont des B-idéaux de* $\mathfrak{S}$ *et si* $\mathfrak{b}\subseteq\mathfrak{a}$ *et* $\mathfrak{a}$ *est un* k_d*-idéal, on a l'isomorphisme*

$$\mathfrak{S}/_{\beta}\mathfrak{a}\simeq(\mathfrak{S}/_{\beta}\mathfrak{b})/_{\beta}(\mathfrak{a}/_{\beta}\mathfrak{b}).$$

Corollaire 2 : *Si* $\mathfrak{a}$ *et* $\mathfrak{b}$ *sont des idéaux d'un demmi-anneau* $\mathfrak{S}$ *à somme commutative, et si* $\mathfrak{b}\subseteq\mathfrak{a}$ *et* $\mathfrak{a}$ *est un k-idéal, on a l'isomorphisme*

$$\mathfrak{S}/_{\beta}\mathfrak{a}\simeq(\mathfrak{S}/_{\beta}\mathfrak{b})/_{\beta}(\mathfrak{a}/_{\beta}\mathfrak{b}).$$

4) Q-idéaux, Q-congruences et morphismes maximaux — Pour pouvoir arriver à un théorème dont l'énoncé est identique au théorème de l'homomorphisme de la Théorie des Anneaux, P. J. Allen [1] a introduit les concepts de Q-idéal et de morphisme maximal pour les demi-anneaux à somme commutative et ayant l'élément-zéro. On trouve dans notre travail [7] des considérations qui étendent déjà les résultats de Allen. En reprenant ici cette question nous améliorons nos propres résultats.

Supposons que l'idéal $\mathfrak{a} \neq \emptyset$ de $\mathfrak{S}$ introduit dans $\mathfrak{S}$ une partition en classes associés à gauche

$$\mathfrak{S}/\xi = \{q + \mathfrak{a} \mid q \in Q \subseteq \mathfrak{S}\}.$$

Alors:

Proposition 1: *$x \in q + \mathfrak{a} \in \mathfrak{S}/\xi$, si et seulement si $x + \mathfrak{a} \subseteq q + \mathfrak{a}$.* En effet: $x \in q + \mathfrak{a}$ implique $x + \mathfrak{a} \subseteq q + \mathfrak{a}$; réciproquement cette condition implique la réciproque.

Corollaire 1: $\forall q \in Q, \quad q \in q + \mathfrak{a}$.

Proposition 2: *Il existe au plus une partition de $\mathfrak{S}$ en classes associées à un idéal $\mathfrak{a}$.* Si $\mathfrak{S}/\xi' = \{q' + \mathfrak{a} \mid q' \in Q' \subseteq \mathfrak{S}\}$ est une deuxième partition de $\mathfrak{S}$, la proposition 1 entraîne $q + \mathfrak{a} \subseteq q' + \mathfrak{a}$, ainsi que $q' + \mathfrak{a} \subseteq q_1 + \mathfrak{a}$, $(q_1 \in Q)$. Alors on a $q + \mathfrak{a} = q' + \mathfrak{a} = q_1 + \mathfrak{a}$.

Proposition 3: *L'ensemble-quotient $\mathfrak{S}/\xi$ est un demi-groupe multiplicatif.* En effet: $(q_1 + \mathfrak{a})(q_2 + \mathfrak{a}) \subseteq q_1 q_2 + \mathfrak{a}$.

Nous étendons notre définition de Q-idéal donnée dans le travail [7], en disant: un *Q-idéal* $\mathfrak{a}$ est un idéal non vide qui définit un demi-anneau-quotient de classes associées à gauche à $\mathfrak{a}$. La congruence correspondante est dite une *Q-congruence* et sera représentée par ξ ou par $\xi_{\mathfrak{a}}$. Le demi-anneau-quotient $\mathfrak{S}\underset{\xi}{/}\mathfrak{a}$ sera appelé un Q-demi-anneau-quotient.

La proposition 1 et la remarque III, de 1, donnent tout de suite:

Proposition 4: *Si $\mathfrak{S}\underset{\xi}{/}\mathfrak{a}$ est un Q-demi-anneau-quotient, toute classe ayant des éléments communs avec $\mathfrak{a}$ est une identité à droite additive.*

Remarques: I) Un demi-anneau admet un Q-idéal singulier $\{z\}$, si et seulement si z est une identité à droite additive et un élément absorbant multiplicatif.

II) Si $\mathfrak{S} \neq \emptyset$ est un G_e-demi-anneau, il n'admet que $\mathfrak{S}$ en tant que Q-idéal.

III) Tout idéal dont les éléments sont des identités à droite additives du demi-anneau est un Q-idéal. En particulier, tous les idéaux non vides d'un G_d-demi-anneau sont des Q-idéaux.

Théorème 1: *Si $\mathfrak{a}$ est un Q-idéal, il en est de même de tout idéal $\mathfrak{a}'$ tel que $\mathfrak{a} \subseteq \mathfrak{a}' \subseteq \bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$, $a \in q_a + \mathfrak{a} \in \mathfrak{S}\underset{\xi}{/}\mathfrak{a}$, et on a*

$$\mathfrak{S}\underset{\xi}{/}\mathfrak{a} = \mathfrak{S}\underset{\xi}{/}\mathfrak{a}' = \mathfrak{S}\underset{\xi}{/} \bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a}).$$

Prenons en effet $q + \mathfrak{a} \in \mathfrak{S}\underset{\xi}{/}\mathfrak{a}$. Alors $q + \mathfrak{a} \subseteq q + \mathfrak{a}' \subseteq q + \bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$. Soit ensuite $x \in \bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$, donc $x \in q_a + \mathfrak{a}$ et $q + x \in q + q_a + \mathfrak{a} \subseteq (q + \mathfrak{a}) + (q_a + \mathfrak{a}) = q + \mathfrak{a}$. On en conclut $q + \mathfrak{a} = q + \mathfrak{a}' = q + \bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$.

Théorème 2: *$\bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$ est le k_d-idéal engendré par $\mathfrak{a}$.* Puisque tous les éléments de l'idéal $\{q_a + \mathfrak{a} \mid a \in \mathfrak{a},\ a \in q_a + \mathfrak{a}\}$ sont des identités à droite additives de $\mathfrak{S}\underset{\xi}{/}\mathfrak{a}$, il s'agit d'un k_d-idéal. $\bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a})$ est aussi un k_d-idéal (en tant qu'image complète inverse d'un k_d-idéal dans l'épimorphisme canonique $\mathfrak{S} \to \mathfrak{S}\underset{\xi}{/}\mathfrak{a}$). Soit $\mathfrak{a}_0$ un k_d-idéal contenant $\mathfrak{a}$. De $a \in q_a + \mathfrak{a}$, on tire $a = q_a + a_1$. Et de $a, a_1 \in \mathfrak{a} \subseteq \mathfrak{a}_0$, on tire $q_a \in \mathfrak{a}_0$ et $q_a + \mathfrak{a} \subseteq \mathfrak{a}_0$, donc $\bigcup_{a \in \mathfrak{a}} (q_a + \mathfrak{a}) \subseteq \mathfrak{a}_0$.

Théorème 3: *La relation de congruence ξ définie par le Q-idéal $\mathfrak{a}$, de $\mathfrak{S}$, implique la relation de congruence ρ, sur $\mathfrak{S}$, si et seulement si toute classe de $\mathfrak{S}/\rho$ qui a des éléments communs avec $\mathfrak{a}$ est une identité à droite additive.* Si $\xi \Rightarrow \rho$, on a l'épimorphisme canonique $\mathfrak{S}\underset{\xi}{/}\mathfrak{a} \to \mathfrak{S}/\rho$, et alors toute classe de $\mathfrak{S}/\rho$ qui a des éléments communs avec $\mathfrak{a}$, puisqu'elle est une image d'une classe de

$\mathfrak{S}/\underset{\xi}{\mathfrak{a}}$ dans les mêmes conditions, est une identité à droite additive.
Réciproquement, en supposant que toute classe D_a de $\mathfrak{S}/\rho$ qui a des éléments communs avec $\mathfrak{a}$ est une identité à droite additive, on a $x \xi y \Rightarrow x = q + a_1$, $y = q + a_2$, $(a_1, a_2 \in \mathfrak{a})$, par conséquent $D_x = D_q + D_{a_1} = D_q$ et $D_y = D_q + D_{a_2} = D_q$, ce qui donne $D_x = D_y$, donc $x \rho y$.

Ce théorème et le théorème 3, n.º 2, donnent les corollaires suivants :

Corollaire 1 : *Si $\mathfrak{a}$ est un Q-idéal et aussi un B-idéal alors on a* $\mathfrak{S}/\underset{\beta}{\mathfrak{a}} = \mathfrak{S}/\underset{\xi}{\mathfrak{a}}$.

Corollaire 2 : *Si $\mathfrak{S}$ est un demi-anneau à somme commutative et $\mathfrak{a}$ est un Q-idéal de $\mathfrak{S}$, la congruence de* Bourne *définie par $\mathfrak{a}$ divise $\mathfrak{S}$ en classes associées à gauche à $\mathfrak{a}$.*

Proposition 5 : *Si l'idéal $\mathfrak{a}$ définit une partition $\mathfrak{S}/\xi$ de $\mathfrak{S}$ en classes associées à gauche à $\mathfrak{a}$, les assertions suivantes sont équivalentes :*

a) $\mathfrak{S}/\xi$ est un demi-anneau-quotient ayant l'élément-zéro (qui évidemment contient $\mathfrak{a}$) ;

a') $\forall q \in Q$, est $\mathfrak{a} + q \subseteq q + \mathfrak{a}$.

D'abord, *a)* $\Rightarrow$ *a')* : Si $\mathfrak{S}/\xi$ contient l'élément-zéro $q_0 + \mathfrak{a}$, on a $\mathfrak{a} \subseteq q_0 + \mathfrak{a}$. D'après la Remarque III, n.º 1, on a aussi $\forall q \in \mathfrak{a}$, $\mathfrak{a} + q \subseteq q + \mathfrak{a}$.
Ensuite, *a')* $\Rightarrow$ *a)* : Pour établir que $\mathfrak{S}/\xi$ est un demi-anneau il ne manque que vérifier qu'on y peut introduire l'opération $+$. Or, on a $(q_1 + \mathfrak{a}) + (q_2 + \mathfrak{a}) = q_1 + (\mathfrak{a} + q_1) + \mathfrak{a} = q_1 + (q_2 + \mathfrak{a}) + \mathfrak{a} \subseteq (q_1 + q_2) + \mathfrak{a}$. Et, si $x \in q + \mathfrak{a}$, de $\mathfrak{a} + x \subseteq \mathfrak{a} + q + \mathfrak{a} \subseteq q + \mathfrak{a} + \mathfrak{a} \subseteq q + \mathfrak{a}$, la Remarque III, n.º 1, assure que toute classe qui a des éléments communs avec $\mathfrak{a}$ est une identité à gauche, donc $\mathfrak{S}/\underset{\xi}{\mathfrak{a}}$ admet un élément neutre pour la somme et cet élément est une classe qui contient $\mathfrak{a}$, par conséquent est l'élément-zéro.

Un *Q_0-idéal* est un idéal dans les conditions de la proposition 5. On fera à propos de ce concept les remarques à suivre.

Remarques : I') Dans un demi-anneau à somme commutative, tout Q-idéal est un Q_0-idéal.
II') Un demi-anneau admet un Q_0-idéal singulier, si et seulement s'il a l'élément-zéro.
III') $\mathfrak{S}$ est un Q_0-idéal, si et seulement s'il existe $q_0 \in \mathfrak{S}$ tel que $q_0 + \mathfrak{S} = \mathfrak{S}$.
IV') Tout Q-idéal contenant un Q_0-idéal est un Q_0-idéal.

Proposition 6 : *Si $\mathfrak{a}$ est un Q_0-idéal de $\mathfrak{S}$, alors $q_0 + \mathfrak{a} \in \mathfrak{S}/\underset{\xi}{\mathfrak{a}}$ tel que $q_0 + \mathfrak{a} \supseteq \mathfrak{a}$ est le k-idéal engendré par $\mathfrak{a}$.*

Corollaire 3 : *Un Q_0-idéal $\mathfrak{a}$ est un k-idéal, si et seulement s'il existe $q_0 \in \mathfrak{S}$ tel que $\mathfrak{a} = q_0 + \mathfrak{a}$.*

Le fait que les résultats pour les Q-idéaux et pour les B-idéaux sont identiques, le fait que dans un demi-anneau à somme commutative tout Q-idéal (d'ailleurs, Q_0-idéal) est un B-idéal (d'ailleurs un B_0-idéal) et l'égalité $\xi_{\mathfrak{a}} = \beta_{\mathfrak{a}}$ lèvent la question de savoir si, dans un demi-anneau quelconque, tout Q-idéal est un B-idéal. La réponse est négative, ce qu'on vérifie en prenant le demi-anneau des tableaux

$+$	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

$\cdot$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	a ,

puisque $\{a, b, c\}$ est un Q_0-idéal $(Q = \{c\})$ mais il n'est pas un B-idéal, car la relation de Bourne qu'il définit nest pas même une relation d'équivalence $(a \beta c$ et $c \beta b$ n'impliquent pas $a \beta b)$.

Théorème 4 : *Si $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ est un épimorphisme de demi-anneaux, $\mathfrak{a}'$ un Q-idéal de $\mathfrak{S}'$ et si $\varphi^{-1}(q' + \mathfrak{a}') = q + \varphi^{-1}(\mathfrak{a}')$, $\forall q' + \mathfrak{a}' \in \mathfrak{S}'/\underset{\xi}{\mathfrak{a}'}$, alors $\varphi^{-1}(\mathfrak{a}') = \mathfrak{a}$ est un Q-idéal, et on a l'isomorphisme $\mathfrak{S}/\underset{\xi}{\mathfrak{a}} \cong \mathfrak{S}'/\underset{\xi}{\mathfrak{a}'}$.*

Si $\mathfrak{S}' = \mathring{\bigcup} (q + \mathfrak{a}')$ est une partition de $\mathfrak{S}'$, on obtient une partition de $\mathfrak{S}$ en posant $\mathfrak{S} = \varphi^{-1}(\mathfrak{S}') = \mathring{\bigcup} \varphi^{-1}(q' + \mathfrak{a}') = \mathring{\bigcup} [q + \varphi^{-1}(\mathfrak{a}')]$. Puis, de $\varphi([q_1 + \varphi^{-1}(\mathfrak{a}')] + [q_2 + \varphi^{-1}(\mathfrak{a}')]) \subseteq (q'_1 + \mathfrak{a}') + (q'_2 + \mathfrak{a}') \subseteq q'_3 + \mathfrak{a}'$, on tire $[q_1 + \varphi^{-1}(\mathfrak{a}')] + [q_2 + \varphi^{-1}(\mathfrak{a}')] \subseteq \varphi^{-1}(q'_3 + \mathfrak{a}') = q_3 + \varphi^{-1}(\mathfrak{a}')$, ce qui achève de prouver que $\varphi^{-1}(\mathfrak{a}')$ est un Q-idéal de $\mathfrak{S}$. L'isomorphisme de l'énoncé est une conséquence immédiate du théorème de l'isomorphisme pour les systèmes algébriques en général, compte tenu de $x \xi_{\mathfrak{a}} y \Leftrightarrow x, y \in q + \varphi^{-1}(\mathfrak{a}') \Leftrightarrow \varphi(x), \varphi(y) \in q' + \mathfrak{a}' \Leftrightarrow \varphi(x) \xi_{\mathfrak{a}'} \varphi(y)$.

Un demi-anneau $\mathfrak{S}$ est dit un Q-demi-anneau (un Q_0-demi-anneau), lorsque son idéal nucléaire $\mathfrak{Z} \neq \emptyset$ est un Q-idéal (un Q_0-idéal). Si $0 \in \mathfrak{S}$, $\mathfrak{S}$ est un Q_0-demi-anneau.

Un épimorphisme $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ de demi-anneaux est appelé maximal, si et seulement si:

i) $\mathfrak{S}'$ est un Q-demi-anneau;
ii) $\varphi^{-1}(q' + \mathfrak{Z}') = q + Ker(\varphi)$, $q' + \mathfrak{Z}' \in \mathfrak{S}' /_{\xi} \mathfrak{Z}'$.

Si $0' \in \mathfrak{S}'$, un épimorphisme $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ est maximal, si et seulement si, $\forall x' \in \mathfrak{S}'$, il existe $x \in \mathfrak{S}$ tel que $\varphi^{-1}(x') = x + Ker(\varphi)$. Il s'agit d'une définition equivalente à celle de ALLEN [1].

THÉORÈME 5: *Si $\mathfrak{S}$ est un demi-anneau dont l'idéal nucléaire $\mathfrak{Z}$ n'est pas vide et si $\mathfrak{a}$ est un Q-idéal, alors l'isomorphisme canonique $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}/\mathfrak{a}$ est maximal.* $\varphi(\mathfrak{Z}) \neq \emptyset$ est l'idéal nucléaire de $\mathfrak{S} /_{\xi} \mathfrak{a}$ et définit sur ce quotient la Q-congruence identité. On a $\varphi^{-1}((q + \mathfrak{a}) + \varphi(\mathfrak{Z})) = \varphi^{-1}(q + \mathfrak{a}) = q + \mathfrak{a}$.

Le Théorème 4 entraîne l'assertion que voici:

THÉORÈME 6: *Si $\mathfrak{S} \xrightarrow{\varphi} \mathfrak{S}'$ est un épimorphisme maximal, $Ker\,\varphi$ est un Q-idéal de $\mathfrak{S}$ et on a*

$$\mathfrak{S} /_{\xi} Ker(\varphi) \cong \mathfrak{S}' /_{\xi} \mathfrak{Z}'.$$

Si l'on se restreint à la classe des demi-anneaux dont les éléments des idéaux nucléaires sont des identités à droite additives (en particulier on considère les demi-anneaux ayant l'élément-zéro), on vérifie qu'il existe un théorème d'homomorphisme analogue à celui de la Théorie des Anneaux, pourvu qu'on se borne aux Q-idéaux et aux épimorphismes maximaux.

BIBLIOGRAPHIE

[1] ALLEN, P. J., *A fundamental theorem of homomorphisms for semirings.* Proc. Am. Math. Soc., **21** (1969), 412-416.

[2] ALMEIDA COSTA, A., *Cours d'Algèbre Générale.* Vol. III. Fundação Calouste Gulbenkian, 1974.

[3] BOURNE, S., *On the homomorphism theorem for semirings.* Proc. Nat. Acad. Sci. U. S. A., **38** (1952), 118-119.

[4] BUGENHAGEN, T. G., *A comparison of three definitions of ideal in a semiring* (paper based on part of the author's doctoral dissertation). University of Tennessee, Knoxville, (1959).

[5] HENRIKSEN, M., *Ideals in semirings with commutative addition.* Am. Math. Soc. Notices, **6** (1965), 321.

[6] LATORRE, D. R., *On h-ideals and k-ideals in hemirings.* Publ. Math. Debrecen, **12** (1965), 219-226.

[7] RAMALHO, M., *Semi-anéis: morfismos e ideais.* Lisboa, (1971).

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BUILDING SEMIRINGS (*)

BY

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To Professor A. ALMEIDA COSTA

In any semiring, the multiplication induces certain multiplicative structures on the family of all Schützenberger groups of the additive semigroup; in turn, these structures determine the multiplication of the semiring to a large extend. These results generalize to Schützenberger groups relative to a congruence contained in the additive Green's relation $\mathcal{D}$, and thus we can use the coextension theory for semigroups to give a complete construction of semirings in terms of groups and mappings [2].

1. MULTIPLICATIVE STRUCTURE ON SCHÜTZENBERGER GROUPS

Let R be a semiring. Clearly its additive Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$ are multiplicative congruences. Let $\mathcal{C}$ be either $\mathcal{D}$ or a congruence of R contained in $\mathcal{D}$, and set $A = R/\mathcal{C}$. Recall that, for every $\mathcal{C}$-class C_α, $\alpha \in A$, of R, if $T_\alpha = \{t \in R^0;\ t + C_\alpha \subseteq C_\alpha\}$, and $\equiv$ is the equivalence relation on T_α defined by: $t \equiv u$ if and only if $t + c = u + c$ for some $c \in C_\alpha$, the *GS group (generalized Schützenberger group)* of C_α is any additive semigroup G_α together with a surjective homomorphism $\pi_\alpha : T_\alpha \to G_\alpha$ which induces the congruence $\equiv$ on T_α; G_α acts simply and transitively on C_α and we denote this action additively, say $\pi_\alpha x + c = x + c$ for all

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$x \in T_\alpha, c \in C_\alpha$. We can show that, for all $\alpha, \beta \in A$, $T_\alpha T_\beta$, $T_\alpha C_\beta$ and $C_\alpha T_\beta$ are all contained in $T_{\alpha\beta}$ and it follows that:

Theorem 1. *For all $\alpha, \beta \in A$, there exists a bihomomorphism $\mu_{\alpha,\beta}: G_\alpha \times G_\beta \to G_{\alpha\beta}$ defined by: $\mu_{\alpha,\beta}(\pi_\alpha t, \pi_\beta u) = \pi_{\alpha\beta}(tu)$ for all $t \in T_\alpha$, $u \in T_\beta$, $\alpha, \beta \in A$.*

We call the family $\mu = (\mu_{\alpha,\beta})_{\alpha,\beta \in A}$ of such mappings a *multiplication* on the family of GS groups $(G_\alpha)_{\alpha \in A}$. Setting $\mu_{\alpha,\beta}(a,b) = a \cdot b$ for all $a \in G_\alpha$, $b \in G_\beta$, $\alpha, \beta \in A$, we have: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a \in G_\alpha$, $b \in G_\beta$, $c \in G_\gamma$, $\alpha, \beta, \gamma \in A$. We express this property by saying that μ is *associative*. Also, from the biadditivity of $\mu_{\alpha,\beta}$ follows easily that $a \cdot b$ and $c \cdot d$ commute in $G_{\alpha\beta}$ for all $a, c \in G_\alpha$, $c, d \in G_\beta$ and $\alpha, \beta \in A$.

Further structure on GS groups is induced by the inner bitranslations of $(R, \cdot)$ as follows.

Theorem 2. *Let $x \in R$, say $x \in C_\alpha$. Then $x T_\beta, T_\beta x \subseteq T_{\alpha\beta}$ for all $\beta \in A$, and there exist homomorphisms $\Lambda^\beta_x: G_\beta \to G_{\alpha\beta}$ and $P^\beta_x: G_\beta \to G_{\beta\alpha}$ defined by: $\Lambda^\beta_x(\pi_\beta u) = \pi_{\alpha\beta}(xu)$, $(\pi_\beta u) P^\beta_x = \pi_{\beta\alpha}(ux)$. Furthermore we have:*

i) $b \cdot (\Lambda^\gamma_x c) = (b P^\beta_x) \cdot c$, $\Lambda^{\beta\gamma}_x(b \cdot c) = \Lambda^\beta_x b \cdot c$, $(b \cdot c) P^{\beta\gamma}_x = b \cdot (c P^\gamma_x)$.

ii) $\Lambda^{\alpha\beta}_y \circ \Lambda^\beta_x = \Lambda^\beta_{yx}$, $P^\beta_x \circ P^{\beta\alpha}_y = P^\beta_{xy}$

for all $x, y \in R$, $x \in C_\alpha$, $b \in G_\beta$, $c \in G_\gamma$, $\alpha, \beta, \gamma \in A$.

Let $\Lambda_x = (\Lambda^\beta_x)_{\beta \in A}$ and $P_x = (P^\beta_x)_{\beta \in A}$ be the families of these homomorphisms. We say that the pair $\Omega_x = (\Lambda_x, P_x)$ is a *bitranslation* on the family of GS groups $(G_\alpha)_{\alpha \in A}$ *induced by x*. Indeed i) is analagous to properties defining for instance bitranslations of semigroups.

More commuting pairs can be produced on $G_{\alpha\beta}$. In fact we have:

Proposition 3. When $x \in C_\alpha$, $y \in C_\beta$, $a, c \in G_\alpha$, $b, d \in G_\beta$, then $\Omega_x d, c\Omega_y$ and $a \cdot b$ all commute whith each other in $G_{\alpha\beta}$. The subgroup $G_{\alpha,\beta}$ of $G_{\alpha\beta}$ generated by all such elements is abelian.

The most important property of the above mappings is the following.

Theorem 4. *For all $x \in C_\alpha$, $a \in G_\alpha$; $y \in C_\beta$, $b \in G_\beta$, $(a + x)(b + y) = (a \cdot b + a P_y + \Lambda_x b) + xy$. Furthermore the three group terms in this formula can be written in any order.*

This result is the basis of the remaining part of the paper, where methods will be given to construct semirings. Note that we cannot expect a precise structure theorem in the case when $\mathcal{C} = \mathcal{H}$, since then no explicit description of the addition is known, so that there is no way to express distributivity conditions in terms of GS groups.

2. CONSTRUCTION OF SEMIRINGS

We now assume that $\mathcal{C}$ is a congruence of R contained in $\mathcal{H}$. Then A is a semiring under the induced operations, and also a category A_L under its right Greens's preorder with A as set of objects and one morphism $\alpha \to \beta$ whenever $\alpha \geqq \beta(\mathcal{R})$, i. e. $\alpha = \beta + \gamma$ for some $\gamma \in A^0$. Then the GS groups of R relative to $\mathcal{C}$ can be organized into a groupvalued functor G, which assigns to each $\alpha \in A$ the group G_α, and to each morphism $\alpha \geqq \beta(\mathcal{R})$ the homomorphism $\varphi^\alpha_\beta: G_\alpha \to G_\beta$ defined by: $\varphi^\alpha_\beta(\pi_\alpha x) = \pi_\beta x$ for all $x \in T_\alpha$ (note that $T_\alpha \subseteq T_\beta$). We call G the left GS functor of R relative to $\mathcal{C}$; we shall also refer to multiplication and bitranslations defined on G instead of $(G_\alpha)_{\alpha \in A}$ above. The main result of [2] then states that, up to isomorphism, $(R, +)$ consists of all pairs (g, α) with $g \in G_\alpha$ and the addition is given by:

$$(a, \alpha) + (b, \beta) = (\varphi^\alpha_{\alpha+\beta} a + \sigma_{\alpha,\beta} + \chi^\beta_{\alpha+\beta} b, \alpha + \beta),$$

where $\sigma_{\alpha,\beta} \in G_{\alpha+\beta}$ and $\chi^\beta_{\alpha+\beta}: G_\beta \to G_{\alpha+\beta}$ is a homomorphism satisfying the following associativity conditions:

(1) $$\sigma_{\alpha,\beta+\gamma} + \chi^{\beta+\gamma}_{\alpha+\beta+\gamma} \sigma_{\beta,\gamma} = \varphi^{\alpha+\beta}_{\alpha+\beta+\gamma} \sigma_{\alpha,\beta} + \sigma_{\alpha+\beta,\gamma};$$

(2) $$\sigma_{\alpha,\beta+\gamma} + \chi^{\beta+\gamma}_{\alpha+\beta+\gamma} \varphi^\beta_{\alpha+\beta} b - \sigma_{\alpha,\beta+\gamma} = \varphi^{\alpha+\beta}_{\alpha+\beta+\gamma} \sigma_{\alpha,\beta} + \varphi^{\alpha+\beta}_{\alpha+\beta+\gamma} \chi^\beta_{\alpha+\beta} b - \varphi^{\alpha+\beta}_{\alpha+\beta+\gamma} \sigma_{\alpha,\beta},$$

for all $\alpha, \beta, \gamma \in A$, $b \in G_\beta$. Let now $p_\alpha = (0, \alpha)$, $\Omega_\alpha b = \Lambda_{p_\alpha} b$, $b\,\Omega_\alpha = b\,P_{p_\alpha}$, $p_\alpha\, p_\beta = \tau_{\alpha,\beta} + p_{\alpha\beta}$ for all $\alpha, \beta \in A$, $b \in G_\beta$. Then it follows easily from Theorem 1 that the multiplication of R is given by:

$$(a, \alpha)(b, \beta) = (a \cdot b + a\,\Omega_\beta + \Omega_\alpha b + \tau_{\alpha,\beta}, \alpha\beta)$$

for all $a \in G_\alpha$, $b \in G_\beta$, $\alpha, \beta \in A$. We can then analize the associativity of the multiplication and distributivity; this yields the following necessary and sufficient conditions:

Associativity conditions:

(3) $\cdot$ is an associative multiplication on G;

(4) Ω_α is a bitranslation on G for all $\alpha \in A$, $\Omega_\alpha (b\,\Omega_\gamma) = (\Omega_\alpha b)\,\Omega_\gamma$;

(5) $\Omega_\alpha \Omega_\beta c = \tau_{\alpha,\beta} \cdot c + \Omega_{\alpha\beta} c$, $c\,\Omega_\alpha \Omega_\beta = c \cdot \tau_{\alpha,\beta} + c\,\Omega_{\alpha\beta}$;

(6) $\Omega_\alpha \tau_{\beta,\gamma} + \tau_{\alpha,\beta\gamma} = \tau_{\alpha,\beta} \Omega_\gamma + \tau_{\alpha\beta,\gamma}$;

(7) the subgroup $G_{\alpha,\beta}$ of $G_{\alpha\beta}$ generated by all $a \cdot b$, $a\,\Omega_\beta$, $\Omega_\alpha b$ with $a \in G_\alpha$, $b \in G_\beta$ is abelian.

Distributivity conditions:

(8) $\varphi^\alpha_{\gamma\delta}(a \cdot b) = \varphi^\alpha_\gamma a \cdot \varphi^\beta_\delta b$, $\varphi^{\alpha\beta}_{\alpha\delta}(\Omega_\alpha b) = \Omega_\alpha (\varphi^\beta_\delta b)$, $\varphi^{\alpha\beta}_{\gamma\beta}(a\,\Omega_\beta) = (\varphi^\alpha_\gamma a)\,\Omega_\beta$ whenever $a \in G$, $b \in G_\beta$, $\alpha \geqq \gamma(\mathscr{R})$, $\beta \geqq \delta(\mathscr{R})$.

(9) $\chi^{\alpha\beta}_{\gamma\delta}(a \cdot b) = k_{\alpha,\beta,\gamma,\delta} + \chi^\alpha_\gamma a \cdot \chi^\beta_\delta b - k_{\alpha,\beta,\gamma,\delta}$,

(9′) $\chi^{\alpha\beta}_{\alpha\delta}(\Omega_\alpha b) = k_{\alpha,\beta,\alpha,\delta} + \Omega_\alpha (\chi^\beta_\delta b) - k_{\alpha,\beta,\alpha,\delta}$,

(9″) $\chi^{\alpha\beta}_{\gamma\delta}(a\,\Omega_\beta) = k_{\alpha,\beta,\gamma,\beta} + (\chi^\alpha_\gamma a)\,\Omega_\alpha - k_{\alpha,\beta,\gamma,\beta}$, whenever $a \in G_\alpha$, $b \in G_\beta$, $\alpha \geqq \gamma(\mathscr{L})$, $\beta \geqq \delta(\mathscr{L})$, where $k_{\alpha,\beta,\gamma,\delta} = \chi^{\alpha\beta}_{\gamma\delta}\tau_{\alpha,\beta} - \tau_{\gamma,\delta}$;

(10) $\Omega_\alpha \sigma_{\beta,\gamma} = \varphi^{\alpha\beta}_{\alpha\beta+\alpha\gamma}\tau_{\alpha,\beta} + \sigma_{\alpha\beta,\alpha\gamma} + \chi^{\alpha\gamma}_{\alpha\beta+\alpha\gamma}\tau_{\alpha,\gamma} - \tau_{\alpha,\beta+\gamma}$;

(10′) $\sigma_{\beta,\gamma}\Omega_\alpha = \varphi^{\beta\alpha}_{\beta\alpha+\gamma\alpha}\tau_{\beta,\alpha} + \sigma_{\beta\alpha,\gamma\alpha} + \chi^{\gamma\alpha}_{\beta\alpha+\gamma\alpha}\tau_{\gamma,\alpha} - \tau_{\beta+\gamma,\alpha}$;

(11) $a \cdot \sigma_{\beta,\gamma} + a\,\Omega_{\beta+\gamma} = \varphi^{\alpha\beta}_{\alpha\beta+\alpha\gamma}(a\,\Omega_\beta) + s_{\alpha,\beta,\gamma} + \chi^{\alpha\gamma}_{\alpha\beta+\alpha\gamma}(a\,\Omega_\gamma) - s_{\alpha,\beta,\gamma}$;

(11′) $\sigma_{\beta,\gamma} \cdot a + \Omega_{\beta+\gamma} a = \varphi^{\beta\alpha}_{\beta\alpha+\gamma\alpha}(\Omega_\beta a) + s'_{\beta,\gamma,\alpha} + \chi^{\gamma\alpha}_{\beta\alpha+\gamma\alpha}(\Omega_\gamma a) - s'_{\beta,\gamma,\alpha}$, where $s_{\alpha,\beta,\gamma} = \varphi^{\alpha\beta}_{\alpha\beta+\alpha\gamma}\tau_{\alpha,\beta} + \sigma_{\alpha\beta,\alpha\gamma}$, $s'_{\beta,\gamma,\alpha} = \varphi^{\beta\alpha}_{\beta\alpha+\gamma\alpha}\tau_{\beta,\alpha} + \sigma_{\beta\alpha,\gamma\alpha}$.

Our main result is then:

THEOREM 5. *Let A be a semiring, G be a group-valued functor on A (preordered by $\geqq(\mathscr{R})$), σ, τ, χ, Ω and $\cdot$ satisfy the above conditions (1) to (11′). Then $R = [G, A; \sigma, \tau, \chi, \Omega, \cdot] = \bigcup_{\alpha \in A}(G_\alpha \times \alpha)$, whith the operations given by:*

$$(a, \alpha) + (b, \beta) = (\varphi^\alpha_{\alpha+\beta} a + \sigma_{\alpha,\beta} + \chi^\beta_{\alpha+\beta} b, \alpha + \beta)$$

$$(a, \alpha)(b, \beta) = (a \cdot b + \Omega_\alpha b + a\,\Omega_\beta + \tau_{\alpha,\beta}, \alpha\beta)$$

is a semiring. Conversely, every semiring R on which there is a congruence $\mathscr{C} \subseteq \mathscr{S}$ such that $R/\mathscr{C} \cong A$, is isomorphic to some $[G, A; \sigma, \tau, \chi, \Omega, \cdot]$.

REFERENCES

[1] A. H. CLIFFORD and G. B. PRESTON, *The algebraic theory of semigroups*. Surveys 7, Amer. Math. Soc., Providence, R. I., 1961, 1967.

[2] P. A. GRILLET, *Left coset extensions*, to appear in the Semigroup Forum

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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FORMAS CASI-FOLIADAS EN VARIEDADES CASI-PRODUCTO COMPLEJO (*)

POR

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Homaxe a o Prof. Dr. ALMEIDA COSTA

La noción de métricas casi-foliadas en una variedad casi-producto real puede extenderse a las variedades casi-producto complejo. Esta noción comprende como caso particular las métricas casi-fibradas (Reinhart [3]).

La métrica hermítica en una variedad casi-compleja se extiende a una pseudo-métrica en la variedad casi-producto complejo inducida. Establecemos para estas pseudométricas la noción de casi-foliadas.

1. FORMAS CASI-FIBRADAS EN VARIEDADES CASI-PRODUCTO COMPLEJO

Una variedad diferenciable M de dimensión n, de clase C^∞, paracompacta se dice casi-producto complejo si se da en el haz fibrado $T^{\mathbb{C}}(M)$, complexificado del haz tangente $T(M)$, una uno-forma vectorial H tal que $H^2=I$ (I, identidad), $H=P-Q$, $P^2=P$, $Q^2=Q$, $PQ=0$; H define dos subfibrados de $T^{\mathbb{C}}(M)$, siendo Tx el espacio tangente a M en el punto $x \in M$, estos subfibrados son:

$T^{1\mathbb{C}}(M)$ cuya fibra en el punto $x \in M$ es $PT_x^{\mathbb{C}}$; y $T^{2\mathbb{C}}(M)$ de fibra $QT_x^{\mathbb{C}} \cdot PT_x^{\mathbb{C}}+QT_x^{\mathbb{C}}=T_x^{\mathbb{C}}$, $T^{1\mathbb{C}}(M)\oplus T^{2\mathbb{C}}(M)=T^{\mathbb{C}}(M)$. ($\oplus$ suma de Whitney).

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Dada en $T^{\mathbb{C}}(M)$ una forma, compleja, lineal simétrica g, compatible con la estructura casi-producto compleja, significa, siendo $\Gamma(T^{\mathbb{C}}(M))$, las secciones C^∞ de $T^{\mathbb{C}}(M)$:

$$g(HX, HY) = g(X, Y), \qquad X, Y \in \Gamma(T^{\mathbb{C}}(M)).$$

La conexión ∇ deducida de g, $(\nabla g) = 0$, la llamaremos primera conexión y llamaremos segunda conexión a la conexión $\tilde{\nabla}$, cuyos coeficientes fueron dados por VAISMAN [4], pero cuya expresión global fue dada por primera vez en [5]:

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{4}\{(\nabla_{HY} H) X + H(\nabla_Y H) X + 2H(\nabla_X H) Y\}.$$

DEFINICIÓN 1

Decimos que la pseudo-métrica g, *definida por la forma compleja bilineal, simétrica* g *es casi-foliada si se verifica*

$$(\tilde{\nabla}_{PX}\, g)(QY, QZ) = 0, \qquad X, Y, Z \in \Gamma(T^{\mathbb{C}}(M)).$$

DEFINICIÓN 2

Decimos que $T^{1\mathbb{C}}(M)$ *es integrable, si siendo* $X, Y \in \Gamma(T^{\mathbb{C}}(M))$,

$$[PX, PY] \in \Gamma(T^{1\mathbb{C}}(M)).$$

Si la variedad M es casi-producto real (M, T^1, T^2), determinada por H dado en $T(M)$ (T^1 de dim. p, T^2 de dim. $n-p$), y se supone T^1 integrable, se establece:

DEFINICIÓN

Una métrica de RIEMANN g *dada en* $(M\, T^1, T^2)$ *diremos que es casi-fibrada, si localmente, en los abiertos de coordenadas* (x, y) *adaptados a la foliación, su expresión es*

$$g = g_{ab}(x, y)\, dx^a\, dx^b + G_{uv}(y)\, dy^u\, dy^v,$$

$$(a, b = 1, 2, \cdots, p; \quad u, v = p+1, \cdots, n).$$

Es equivalente a que localmente existe distancia entre placas. Para las variedades casi-producto reales, se deduce la proposición siguiente [7]:

PROPOSICIÓN 1

Si g *es casi-foliada y* T^1 *es integrable,* g *es métrica casi-fibrada.*

Esta proposición permite definir pseudo-métricas casi-fibradas en las variedades casi-producto complejo.

DEFINICIÓN 3

Una forma g *bilineal, simétrica, dada en una variedad* $(M, T^{\mathbb{C}1}, T^{\mathbb{C}2})$, *se dirá que es una pseudo-métrica casi-fibrada, si* g *es casi-foliada y* $T^{\mathbb{C}1}$ *integrable.*

Se pueden generalizar a las pseudo-métricas casi-fibradas muchas de las propiedades de las métricas casi-fibradas, como por ejemplo los teoremas de MOLINO [2] sobre las conexiones proyectables

2. FORMAS CASI-FOLIADAS EXTENSIÓN DE MÉTRICAS HERMÍTICAS

Dada una variedad diferenciable M, de dimensión $2n$, para-compacta, C^∞, en la que se da en $T(M)$ el operador $J^2 = -I$, se dice casi-compleja, y una métrica h, tal que

$$h(JX, JY) = h(X, Y)$$

se dice métrica hermítica.

J se extiende al complexificado $T^{\mathbb{C}}(M)$ y determina una estructura casi-producto complejo $H = iJ$. También h se extiende al complexificado en una forma compleja, bilineal simétrica [1] que seguiremos representando por h; pero esta forma no está adaptada a la estructura casi-producto, como se deduce de la proposición siguiente:

PROPOSICIÓN 2

$$h(HX, HY) = -h(X, Y), \qquad X, Y \in \Gamma(T^{\mathbb{C}}(M)).$$

Demostración

Es consecuencia de ser

$$h(JX, JY) = h(X, Y).$$

Siendo $\tilde{\nabla}$ la segunda conexión y ∇ la conexión correspondiente a h, se deduce:

Proposición 3

$$(\tilde{\nabla}_{PX} h)(QY, QZ) = 0.$$

Demostración

Basta tener en cuenta

$$(\tilde{\nabla}_{PX} h)(QY, QZ) = (\nabla_{PX} g)(QX, QY) -$$

$$-\frac{1}{4}\Big[h\Big(-(\nabla_{QY} H)PX + H(\nabla_{QY} H)PX + 2H(\nabla_{PX} H)QY, QZ\Big) +$$

$$+ h\Big(QY, -(\nabla_{QZ} H)PX + H(\nabla_{QZ} H)PX + 2H(\nabla_{PX} H)QZ\Big)\Big].$$

Por tanto, en este caso, la definición de métrica casi-foliada no puede ser la dada en la Definición 1. Para deducir propiedades análogas para h a las correspondientes a g, hemos de establecer la definición siguiente:

Definición 4

Dada en una variedad casi-producto complejo $(M, T^{\Phi 1}, T^{\Phi 2})$ *una pseudo-métrica* h, *tal que* $h(HX, HY) = -h(X, Y)$ *diremos que es casi-foliada si se verifica:*

$$(\tilde{\nabla}_{PX} h)(PY, QZ) = 0, \qquad X, Y, Z \in \Gamma(T^{\Phi}(M)).$$

Se deduce inmediatamente

Proposición 4

Si h *es casi-foliada, se verifica:*

$$Q \nabla_{QX} PY = 0 \qquad X, Y \in \Gamma(T^{\Phi}(M)).$$

También en este caso se encuentran pseudo-métricas con propiedades análogas a las métricas casi-fibradas, considerando formas h casi-foliadas y $T^{\Phi 1}$ integrable.

BIBLIOGRAFIA

[1] Kobayashi-Nomizu, *Foundations of differential geometry*. Vol. II. Interscience. New York, 1969.

[2] P. Molino, Comp. Rendus Ac. Sci. Paris, 272, serie A, 1971, p. 779.

[3] B. L. Reinhart, *Foliated manifolds with bundle-metric.* Ann. of Math. **69** (1959), 119-132.

[4] I. Vaisman, *Variétés riemanniennes feuilletées.* Czeckoslovak Math. J., **21** (96), 46-75 (1971).

[5] E. Vidal, Comp. rendus Ac. Sci. Peris, 273, serie A, 1971, p. 1152.

[6] E. Vidal, Comp. rendus Ac. Sci. Paris, 277, serie A, 1973, p. 461.

[7] E. Vidal—E. Vidal Costa, *Special connections and almost foliated metrics.* Jour. of diff. geometry **8** (1973), 297-304.

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ESPAÑA

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R-ÁLGEBRAS ANALÍTICAS VALORADAS(*)

POR

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Al Prof. ALMEIDA COSTA, en su septuagésimo aniversário, mui cordialmente.

En el estudio de los conjuntos semianalíticos [1], [2], se observa una estructura algebraica subyacente que podría simplificar algunas demostraciones. Esta es la finalidad de la construcción y estudio de las R-Algebras Analíticas Valoradas. Naturalmente que, al introducir la estructura algebraica, se pierden los conjuntos semianalíticos abiertos, pero la estructuración alcanzada para el estudio de los conjuntos semianalíticos cerrados estimamos que compensa esta pérdida. En este artículo estudiaremos la estructura de dichas R-Algebras, dejando para otro lugar su aplicación a la teoría de conjuntos semianalíticos y subanalíticos [2] cerrados.

1. R-ALGEBRAS ANALÍTICAS VALORADAS.

Sea A la R-álgebra de las funciones analíticas reales, definidas sobre un abierto conexo Ω de R^n, y sea $B = A[X_1, \cdots, X_r]$ el álgebra de polinomios con coeficientes de A en las indeterminadas: $X_1, \cdots, X_r$, sobre A. B se puede considerar también como un anillo de funciones analíticas sobre $\Omega \times R^r$, que es también un conjunto conexo.

Es bien conocido y, por otra parte, de comprobación inmediata, el siguiente:

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Lema 1. *Toda* R-*álgebra de funciones analíticas sobre un abierto conexo es un dominio de integridad.*

Lema 2. *Los ideales:*

$$B(X_1 - f_1, \cdots, X_s - f_s), \quad s \leq r, \quad f_i \in A, \quad i = 1, \cdots, s,$$

son primos.

Demostración. Todo elemento $F(X_1, \cdots, X_s) \in B$ se puede escribir en la siguiente forma:

$$F(X_1, \cdots, X_r) = G(X_1 - f_1, \cdots, X_s - f_s, X_{s+1}, \cdots, X_r)$$
$$= G_1(X_1 - f_1, \cdots, X_s - f_s, X_{s+1}, \cdots, X_r)(X_1 - f_1)$$
$$+ G'_1(X_2 - f_2, \cdots, X_s - f_s, X_{s+1}, \cdots, X_r),$$

siendo G'_1 un polinomio en el que no figura la variable X_1. Repitiendo el mismo proceso con G'_1, y así siguiendo, se obtiene:

$$F = G_1(X_1 - f_1) + \cdots + G_s(X_s - f_s) + G',$$

en donde G' es un polinomio en las variables $X_{s+1}, \cdots, X_r$, con coeficientes en A.

Si F y H son elementos de B tales que:

$$F H \in B(X_1 - f_1, \cdots, X_s - f_s)$$

y si

$$H = L_1(X_1 - f_1) + \cdots + L_s(X_s - f_s) + L',$$

en donde L' es un polinomio en las variables $X_{s+1}, \cdots, X_r$, de las relaciones anteriores se deduce que

$$G' \cdot L' \in B(X_1 - f_1, \cdots, X_s - f_s),$$

esto es

$$G' \cdot L' = P_1(X_1 - f_1) + \cdots + P_s(X_s - f_s),$$

de donde, haciendo $X_i = f_i$, $i = 1, \cdots, s$, y recordando que G' y L' no dependen de estas variables, resulta que

$$G' \cdot L' = 0$$

y como G' y L' son funciones analíticas sobre el conjunto conexo $\Omega \times R^r$, resulta, en virtud del lema anterior, que uno de ambos polinomios es cero, con lo que queda probado el lema.

Designamos por I_j al conjunto $\{1, -1\}^j$. Si $\alpha = (\alpha_1, \cdots, \alpha_t) \in I_t$, pondremos: $J_\alpha = B(X_1 - \alpha_1 f_1, \cdots, X_1 - \alpha_t f_t)$, siendo $t \leq r$ y $\{f_1, \cdots, f_r\}$ un conjunto de funciones de A fijado para todo lo que sigue.

Lema 3.

$$\bigcap_{i \in I_t} (J_i + B(X_u^2 - f_u^2)) = \left(\bigcap_{i \in I_t} J_i \right) + B(X_u - \iota_u), \quad 0 < t < u \leq r.$$

Demostración. Si $Q \in J_i$ pondremos, abreviadamente:

$$Q = A_{i1}(X_1 - i_1 f_1) + \cdots + A_{it}(X_t - i_t f_t) = A_i(X - i f), \quad i \in I_t.$$

Para todo

$$P \in \bigcap_{i \in I_t} (J_i + B(X_u^2 - f_u^2)) \text{ se verifica que:}$$

(1) $$P = P_i + C_i(X_u^2 - f_u^2), \quad P_i \in J_i, \quad i \in I_t.$$

Por tanto,

$$P_i = A_i(X - i f),$$

y los polinomios A_{ij} se puede escribir en la forma:

$$A_{ij} = A'_{ij} + X_u A''_{ij} + (X_u^2 - f_u^2) A'''_{ij},$$

en donde A'_{ij} y A''_{ij} son polinomios en los que no figura la variable X_u, por consiguiente:

$$P_i = (A'_i + X_u A''_i)(X - i f) + (X_u^2 - f_u^2) A'''_i$$

y, sustituyendo en (1)

(2) $$P = (A'_i + X_u A''_i)(X - i f) + D_i(X_u^2 - f_u^2),$$

$$D_i = A'''_i + C_i, \quad i \in I_t.$$

Como el primer miembro de (2) no depende de i, resulta que $\forall\, i, j \in I_t$,

$$(A'_i + X_u A''_i)(X - i\,f) + D_i(X_u^2 - f_u^2) =$$
$$= (A'_j + X_u A''_j)(X - j\,f) + D_j(X_u^2 - f_u^2),$$

y, haciendo $X_u = f_u$ y a continuación $X_u = -f_u$, se obtiene:

$$(A'_i + f_u A''_i)(X - i\,f) = (A'_j + f_u A''_j)(X - j\,f)$$
$$(A'_i - f_u A''_i)(X - i\,f) = (A'_j - f_u A''_j)(X - j\,f)$$

de donde

$$A'_i(X - i\,f) = A'_j(X - j\,f)$$

y

$$f_u A''_i(X - i\,f) = f_u A''_j(X - j\,f)$$

y como tanto f_u como las restantes funciones que figuran en la igualdad precedente son analíticas en un conjunto conexo, y $f_u \neq 0$ en él, resulta:

$$A''_i(X - i\,f) = A''_j(X - j\,f),$$

$(A'_i + X_u A''_i)(X - i\,f) \in \bigcap_{i \in I_t} J_i$ y, por tanto, en virtud de (2),

$$P \in \Big(\bigcap_{i \in I_t} J_i\Big) + B(X_u^2 - f_u^2),$$

esto es, se ha probado la inclusión $\subset$. Como la inclusión $\supset$ es obvia, queda probado el lema.

LEMMA 4. $\bigcap_{i \in I_s} J_i = B(X_1^2 - f_1^2, \cdots, X_s^2 - f_s^2)$, $1 \leq s \leq r$.

DEMOSTRACIÓN. La inclusión $\supset$ es trivial.

Para probar la otra inclusión empleamos inducción respecto de s. Para $s = 1$, si $F \in B(X_1 - f_1) \cap B(X_1 + f_1)$, será

$$F = F_1(X_1 - f_1) \in B(X_1 + f_1)$$

y como, por el lema 2, el ideal $B(X_1 + f_1)$ es primo, uno de los dos factores pertenecerá a este ideal. Ahora bien si $X_1 - f_1 \in B(X_1 + f_1)$ será

$$X_1 - f_1 = P(X)(X_1 + f_1)$$

y haciendo $X_1 = -f_1$, resulta $f_1 = 0$, contradicción. Por tanto, $F_1 \in B(X_1 + f_1)$, esto es, $F_1 = Q(X)(X_1 + f_1)$ y $F = Q(X_1 - f_1) \cdot (X_1 + f_1)$.

Supuesta demostrada la inclusión $\subset$ para $s \leq t$, sea

$$F \in \bigcap_{i \in I_{t+1}} J_i,$$

esto es

$$F = F_i(X - i\,f), \quad i \in I_{t+1}.$$

Considerando, en particular,

$$i = (i_1, \cdots, i_t, 1), \quad j = (i_1, \cdots, i_t, -1),$$

resulta que

$$F_{i,t+1}(X_{t+1} - f_{t+1}) \in J_j.$$

Si $X_{t+1} - f_{t+1}$ perteneciese a J_j, sería

$$X_{t+1} - f_{t+1} = E_1(X_1 - i_1 f_1) + \cdots +$$
$$+ E_t(X_t - i_t f_t) + E_{t+1}(X_{t+1} + f_{t+1})$$

y haciendo $X_j = i_j f_j$, $j = 1, \cdots, t$, y $X_{t+1} = -f_{t+1}$, se obtiene $f_{t+1} = 0$, contradicción, luego $X_{t+1} - f_{t+1} \notin J_j$ y como este ideal es primo, resulta:

$$F_{i,t+1} \in J_j,$$

esto es,

$$F_{i,t+1} = F'_1(X_1 - i_1 f_1) + \cdots + F'_t(X_t - i_t f_t) + F'_{t+1}(X_{t+1} + f_{t+1}),$$

luego,

$$F = M_1(X_1 - i_1 f_1) + \cdots + M_t(X_t - i_t f_t) +$$
$$+ F'_{t+1}(X_{t+1}^2 - f_{t+1}^2), \quad i \in I_t,$$

esto es:

$$F \in \bigcap_{i \in I_t} [J_i + B(X_{t+1}^2 - f_{t+1}^2)]$$

y, por el L. 3,

$$F \in \Big(\bigcap_{i \in I_t} J_i\Big) + B(X_{t+1}^2 - f_{t+1}^2),$$

pero, por la hipótesis de inducción y la inclusión $\supset$ válida para todo s, se verifica que

$$\bigcap_{i \in I_t} J_i = B(X_1^2 - f_1^2, \cdots, X_t^2 - f_t^2),$$

con lo que queda probada la inclusión $\subset$ para $t+1$, y por tanto el lema.

Escolio 1. Para todo $F \in B$ y para todo $i \in I_r$, se verifica que:

$$F = F_1(X_1 - i_1 f_1) + \cdots + F_r(X_r - i_r f_r) + f, \quad f \in A.$$

Notación. Sea $L \subset I_r = \{1, -1\}^r$ el conjunto de indices tal que para todo $i \in L$ se verifique que el interior de V_i sea distinto del conjunto vacio, y recíprocamente. Se designará por F_i el conjunto de todos los puntos $x \in \Omega$ tales que $f_i(x) = 0$.

Lema 5. $\ker u = \bigcap_{i \in L} J_i$.

Demostración. Sea $J = \bigcap_{i \in L} J_i$.

(i) Para todo $F \in J$, sea $F' = u(F)$. Para todo punto $x \in \Omega$ tal que $x \notin \bigcup_{i=1}^{r} F_i$, existe un $i \in L$ tal que $x \in V_i$ y como $F \in J \subset J_i$, será $F'(x) = 0$. Por consiguiente, $F'(x) = 0$ para todo punto x de $\Omega - \bigcup_{i=1}^{r} F_i$ y como $\bigcup_{i=1}^{r} F_i$ es un conjunto raro en Ω y como el conjunto de los puntos en que se anula F' es cerrado, por ser F' función continua, resulta que F' se anula en Ω, esto es, F' es la función cero sobre Ω y, por tanto, $F \in \ker u$.

(ii) Para todo $F \in B$ y para todo $i \in L$, se puede escribir:

$$F = \sum_{k=1}^{r} g_{k_i}(X_k - i_k f_k) + h_i, \tag{3}$$

en donde $g_{k_i} \in B$, $k = 1, \cdots, r$, y $h_i \in A$, $i \in L$.

Si $F \in \ker u$, será:

$$0 = u(F) = \sum_{k=1}^{r} u(g_{k_i})(|f_k| - i_k f_k) + h_i,$$

de donde, en virtud del L. 1 de 2.,

$$\sum_{k=1}^{r} u(g_{k_i})(|f_k| - i_k f_k) = -h_i \in I_i \cap A = \{0\},$$

y, teniendo en cuenta (3),

$$F = \sum_{k=1}^{r} g_{k_i}(X_k - i_k f_k) \in J_i,$$

para todo $i \in L$, luego $F \in J$.

Escolio 2. $u\Big(\bigcap_{i \in L} J_i\Big) = \bigcap_{i \in L} u(J_i)$.

En effecto, la inclusión $\subset$ es obvia. Para todo $F' \in \bigcap_{i \in L} u(J_i)$ se verifica que:

$$F' = \sum_{j=1}^{r} g_{ij}(|f_j| - i_j f_j) = \sum_{j=1}^{r} g_{k_j}(|f_j| - k_j f_j); \; i, k \in L. \tag{4}$$

Sea

$$F_i = \sum_{j=i}^{r} g_{i_j}(X_j - i_j f_j). \tag{5}$$

De (4) y (5) se deduce que

$$u(F_i - F_j) = 0\,,\quad F_i - F_j \in \ker u = \bigcap_{i \in L} J_i\,,$$

luego,

$$F_i = F_j + F_{ij} \in J_j\,,\quad F_{ij} \in \bigcap_{i \in L} J_j\,,\quad \forall j \in L\,,$$

luego,

$$F_i \in \bigcap_{j \in L} J_j\,,$$

luego,

$$F' = u(F_i) \in u\Big(\bigcap_{j \in L} J_j\Big)\,.$$

Escolio 3. $\{0\} = \bigcap_{j \in L} I_j$.

Escolio 4. Los ideales I_j, $j \in L$, son primos.

Se ha obtenido el siguiente:

Teorema 1. *En las hipótesis anteriores, se verifica que:*

$$0 \to \bigcap_{i \in L} \mathrm{J}_i \to \mathrm{A}[\mathrm{X}_1, \cdots, \mathrm{X}_r] \xrightarrow{u} \mathrm{A}[\,|\mathrm{f}_1|, \cdots, |\mathrm{f}_r|\,] \to 0\,,$$

es una sucesión exacta.

2. RELACIONES ENTRE LOS IDEALES DE A y $\overline{\mathrm{A}}$

Si $V_j = \{x \in \Omega \mid G(x) = 0,\ \forall\, G \in I_j\}$, se verifica que existe algún $j \in I_r$ tal que $\mathring{V}_j \neq \phi$, ya que $\bigcup_{j \in I_r} V_j = \Omega$ y $\bigcup_{i=1}^{r} F_i$ es un conjunto raro en Ω.

Lema 1. *Si* $\mathring{\mathrm{V}}_j \neq \phi$ *se verifica que* $\mathrm{I}_j \cap \mathrm{A} = \{0\}$.

Demostración. Si $F \in I_j \cap A$ y $V(F) = \{x \in \Omega \mid F(x) = 0\}$, de $\{F\} \subset I_j$ se deduce que $V(F) \supset V_j$, luego $V(F)$ contiene un abierto no vacio de Ω y, por ser Ω conexo y F analítico sería $F = 0$ sobre Ω.

Lema 2. *Si* m *es un ideal de* A, *se verifica que*

$$\overline{\mathrm{A}}\mathrm{m} \cap \mathrm{A} = \mathrm{m}\,.$$

Demostración. Evidentemente es $\mathrm{m} \subset \overline{A}\mathrm{m} \cap A$. Para todo $F \in \overline{A}\mathrm{m} \cap A$ y para todo I_j tal que $\mathring{V}_j \neq \phi$, se verifica que:

$$F = \sum_{i=1}^{i=s} H_i p_i\,,\quad H_i \in \overline{A}\,,\quad p_i \in \mathrm{m}\,,\quad i = 1, \cdots, s\,.$$

Ahora bien, H_i se puede escribir en la forma:

$$H_i = H'_i + h_i\,,\quad H'_i \in I_j\,,\quad h_i \in A\,,$$

luego,

$$F = \sum_{i=1}^{i=s} H'_i p_i + \sum_{i=1}^{i=s} h_i p_i\,,$$

de donde,

$$F - \sum_{i=1}^{i=s} h_i p_i = \sum_{i=1}^{i=s} H'_i p_i \in I_j \cap A = \{0\}\,,$$

esto es, $F = \sum_{i=1}^{i=s} h_i p_i \in \mathrm{m}$.

Escolio. $(I_j + \overline{A}\mathrm{m}) \cap A = \mathrm{m}$, siendo $\mathring{V}_j \neq \phi$.

Lema 3. *Si* p *es un ideal primo de* A *y* $\mathring{\mathrm{V}}_j \neq \phi$, *se verifica que*

$$\mathrm{P} = \mathrm{I}_j + \overline{\mathrm{A}}\,\mathrm{p}$$

es ideal primo de $\overline{\mathrm{A}}$.

Demostración. Sean $F, G \in P$. Poniendo $F = F' + f$, $G = G' + g$, $F', G' \in I_j$; $f, g \in A$, resulta que

$$FG = H + fg, \quad H \in I_j.$$

Pero, por hipótesis,

$$FG = H' + \sum_{i=1}^{i=t} L_i p_i, \quad L_i \in \overline{A}, \quad p_i \in \mathrm{p},$$

y poniendo $L_i = L'_i + l_i$, $L'_i \in I_j$, $l_i \in A$, $i = 1, \cdots, t$, resulta:

$$H + fg = H' + \sum_{i=1}^{t} L'_i p_i + \sum_{i=1}^{t} l_i p_i,$$

de donde

$$H - H' - \sum_{i=1}^{t} L'_i p_i = \sum_{i=1}^{t} l_i p_i - fg \in I_j \cap A = \{0\},$$

luego

$$fg = \sum_{i=1}^{t} l_i p_i \in \mathrm{p}$$

y, por ser p primo, uno, por lo menos de los dos factores pertenece a p. Si $f \in \mathrm{p}$ será $F = F' + f \in I_j + \overline{A}\mathrm{p} = P$.

Corolario. *Si* q *es un ideal primario de* A *y* $\sqrt{\mathrm{q}} = \mathrm{p}$, *se verifica que*

$\mathrm{I_j} + \mathrm{A\,q}$ *es un ideal primario y* $\sqrt{\mathrm{I_j} + \mathrm{A\,q}} = \mathrm{I_j} + \overline{\mathrm{A}}\mathrm{p}$, *siendo* $\mathring{\mathrm{V}}_{\mathrm{j}} \neq \phi$.

Lema 4. *Si* P *es un ideal primo de* $\overline{\mathrm{A}}$ *y* $\mathrm{p} = \mathrm{P} \cap \mathrm{A}$, *existe un* $\mathrm{k} \in \mathrm{I_r}$ *tal que*

$$\mathrm{P} = \mathrm{I_k} + \overline{\mathrm{A}}\mathrm{p}.$$

Demostración. De $\{0\} = \bigcap_{i \in I} I_i \subset P$ y de ser los ideales I_i primos resulta que P divide a uno, por lo menos, de los ideales I_i.

Sea $I_k \subset P$. Por tanto, $I_k + \overline{A}\mathrm{p} \subset P$. Para todo $F \in P$ se verifica que

$$F = H + h, \quad H \in I_k, \quad h \in A,$$

luego $H \in P$ y, por tanto, $F - H = h \in P$, luego, $h \in \mathrm{p}$.

Lema 5. *Si* $\mathrm{q_i}$, $\mathrm{i} = 1, \cdots, \mathrm{r}$ *son ideales de* A *y si* $\mathrm{I_j}$ *es tal que* $\mathring{\mathrm{V}}_{\mathrm{j}} \neq \phi$, *se verifica que*

$$\mathrm{I_j} + \overline{\mathrm{A}}(\mathrm{q_1} \cap \cdots \cap \mathrm{q_r}) = \mathrm{I_j} + (\overline{\mathrm{A}}\,\mathrm{q_1} \cap \cdots \cap \overline{\mathrm{A}}\,\mathrm{q_r}).$$

Demostración. La inclusión $\subset$ es obvia. Para todo

$$F \in I_j + (\overline{A}\mathrm{q_1} \cap \cdots \cap \overline{A}\mathrm{q_r}),$$

se verifica que

$$F = H_1 + h_1 = \cdots = H_r + h_r, \quad H_i \in I_j, \quad h_i \in \mathrm{q}_i, \quad i = 1, \cdots, r.$$

Por tanto,

$$H_k - H_l = h_l - h_k \in I_j \cap A = \{0\},$$

luego,

$$h_1 \in \bigcap_{i=1}^{r} \mathrm{q}_i$$

y

$$F = H_1 + h_1 \in I_j + A(\mathrm{q_1} \cap \cdots \cap \mathrm{q_r}).$$

Lema 6. *En las mismas hipótesis del lema anterior,*

$$(\mathrm{I_j} + \overline{\mathrm{A}}\,\mathrm{q_1}) \cap \cdots \cap (\mathrm{I_j} + \overline{\mathrm{A}}\,\mathrm{q_r}) = \mathrm{I_j} + (\overline{\mathrm{A}}\,\mathrm{q_1} \cap \cdots \cap \overline{\mathrm{A}}\,\mathrm{q_r}).$$

La inclusión $\supset$ es obvia. Para todo

$$F \in (I_j + \overline{A}\mathrm{q_1}) \cap \cdots \cap (I_j + \overline{A}\mathrm{q_r})$$

se verifica que

$$F = H_1 + h_1 = \cdots = H_r + h_r, \quad H_i \in I_j, \quad h_i \in \mathrm{q}_i, \quad i = 1, \cdots, r,$$

de donde, como en el lema anterior,

$$F \in I_j + (\bar{A}q_1 \cap \cdots \cap \bar{A}q_r).$$

Lema 7. *Si* I_j *es tal que* $V_j \neq \phi$ *y* m *es un ideal de* A, *se verifica que*

$$(I_j + A\,m) \cap A = m.$$

En efecto, si $f \in (I_j + \bar{A}m) \cap A$, se verifica que

$$f = H + h, \quad H \in I_j, \quad h \in m,$$

luego $f - h = H \in I_j \cap A = \{0\}$, y $f = h \in m$.
De los lemas anteriores resulta la siguiente:

Proposición. *Si*

$$m = q_1 \cap \cdots \cap q_r$$

es un ideal de A, *siendo* q_i, $i = 1, \cdots, r$, *ideales primarios y* $\sqrt{q_i} = p_i$, *se verifica que, para todo* I_j *tal que* $\mathring{V}_j \neq \phi$,

$$I_j + \bar{A}(q_1 \cap \cdots \cap q_r) = (I_j + \bar{A}q_1) \cap \cdots \cap (I_j + \bar{A}q_r)$$

en donde los ideales $I_j + \bar{A}q_k$, $k = 1, \cdots, r$, son primarios yacen sobre q_k y

$$\sqrt{I_j + \bar{A}q_k} = I_j + \bar{A}p_k, \quad k = 1, \cdots, r.$$

REFERENCIAS

[1] S. Lojasiewicz, *Ensembles Semi-Analytiques*, Centre de Physique théorique de l'École Polytechnique. (Policopié). 1965.

[2] H. Hironaka, *Subanalytic sets. Number Theory, Algebraic Geometry, and Commutative Algebra*, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 453-493.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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A COMMON FIXED POINT THEOREM OF CONTRACTION MAPPINGS (*)

BY

KIYOSHI ISÉKI

Dedicated to Prof. A. ALMEIDA COSTA in deep esteem

Recently many papers related to the contraction theorem by S. Banach have been published (I. A. Rus [3] contains a detail bibliography).

Some authors have investigated under what conditions two mappings f, g on a complete metric space X have a unique common fixed point. In particular, I. A. Rus (2) has obtained a general result which is generalization by R. Kannan and S. K. Chatterjea theorems. On the other hand, R. E. DeMarr [1] obtained common fixed point theorem for a family of mappings.

In this paper, we prove following

Theorem. *Let* (X, ρ) *be a complete metric space, and* f, g *two continuous mappings on* X *into itself. If there are positive numbers* α, β *satisfying* α, $\beta < 1$ *such that, for all* $x, y \in X$,

(1) $$\rho\big(f(g(x)), g(y)\big) \leq \alpha\,\rho(x, g(y)),$$

(2) $$\rho\big(g(f(x)), f(y)\big) \leq \beta\,\rho(x, f(y)),$$

then f *and* g *have a unique common fixed point, i.e. there is a unique element* x_0 *such that* $f(x_0) = g(z_0) = x_0$.

Proof. Let h and k be composed mappings fg and gf respectively. Let x be any element of X. For the element x, and for natural number m, n such that $m < n$,

(*) Recebido em 14 de Janeiro de 1974.

$$(3)\qquad \rho(h^n(x),\ h^m(x)) \leqq \rho(h^n(x),\ g h^{n-1}(x))$$
$$+\rho(g h^{n-1}(x),\ h^{n-1}(x)) + \rho(h^{n-1}(x),\ g h^{n-2}(x)) + \cdots$$
$$+\rho(h^{m+1}(x),\ g h^m(x)) + \rho(g h^m(x),\ h^m(x)).$$

By using the given conditions (1), (2), for any natural number l, we have

$$(4)\qquad \rho(h^l(x),\ g h^{l-1}(x)) \leqq \alpha^l \beta^{l-1} \rho(x,\ g(x)),$$

$$(5)\qquad \rho(g h^l(x),\ h^l(x)) \leqq \alpha^l \beta^l \rho(x,\ g(x)).$$

The proofs are due to induction. For $l=1$, the first inequality is

$$\rho(h(x),\ g(x)) \leqq \alpha\rho(x,\ g(x)),$$

which is (1). From (1), (2), we have

$$\rho(gfg(x);\ fg(x)) \leqq \beta\rho(g(x);\ fg(x)) \leqq \alpha\beta\rho(x,\ g(x)),$$

which is (5) for $l=1$. Moreover

$$\begin{aligned}
&\rho(h^{l+1}(x),\ g h^l(x))\\
&= \rho(h^l(h(x)),\ g h^{l-1}(h(x)))\\
&\leqq \alpha^l \beta^{l-1} \rho(h(x),\ g h(x))\\
&\leqq \alpha^l \beta^{l-1} \rho(fg(x),\ gfg(x))\\
&\leqq \alpha^l \beta^l \rho(g(x),\ fg(x))\\
&\leqq \alpha^{l+1} \beta^l \rho(x,\ g(x)),
\end{aligned}$$

which completes the proof of (4). Concerning (5), we have

$$\begin{aligned}
&\rho(g h^{l+1}(x),\ h^{l+1}(x))\\
&= \rho(g h^l h(x),\ h^l h(x))\\
&= \alpha^l \beta^l \rho(h(x),\ g h(x))\\
&= \alpha^l \beta^l \rho(fg(x),\ gfg(x))\\
&\leqq \alpha^l \beta^{l+1} \rho(g(x),\ fg(x))\\
&\leqq \alpha^{l+1} \beta^{l+1} \rho(x,\ g(x)),
\end{aligned}$$

which completes of the proof of (5).

From (3), (4), (5), we have

$$\begin{aligned}
\rho(h^n(x),\ h^m(x)) &\leqq (\alpha^n\beta^{n-1} + \alpha^{n-1}\beta^{n-1} + \alpha^{n-1}\beta^{n-2} + \cdots +\\
&\qquad \alpha^{m+1}\beta^m + \alpha^m\beta^m)\rho(x,\ g(x))\\
&\leqq (\alpha^n\beta^{n-1} + \alpha^{n-1}\beta^{n-2} + \cdots + \alpha^{m+1}\beta^m)\rho(x,\ g(x))\\
&\quad + (\alpha^{n-1}\beta^{n-1} + \alpha^{n-2}\beta^{n-2} + \cdots + \alpha^m\beta^m)\rho(x,\ g(x)).
\end{aligned}$$

By the hypothesis, we have $\alpha\beta<1$, and then

$$\rho(h^n(x),\ h^m(x)) \to 0 \qquad (n, m \to \infty),$$

hence by the completeness of X, there is an element x_0 which is the limit of $\{h^n(x)\}$.

From the continuity of f and g, $h^n(x) \to x_0 (n\to\infty)$ implies $h^n(x) \to fg(x_0)\ (n\to\infty)$.

On the other hand, we have

$$\begin{aligned}
\rho(x_0, g(x_0)) &\leqq \rho(x_0, h^n(x)) + \rho(h^n(x),\ fg(x_0))\\
&\qquad + \rho(fg(x_0),\ g(x_0))\\
&\leqq \rho(x_0, h^n(x)) + \rho(h^n(x),\ fg(x_0)) + \alpha\rho(x_0, g(x_0)).
\end{aligned}$$

Therefore $n\to\infty$ implies

$$\rho(x_0, g(x_0)) \leqq \alpha\rho(x_0, g(x_0)),$$

which means $\rho(x_0, g(x_0))=0$, i. e. $g(x_0)=x_0$. Hence x_0 is a fixed point of g. For this element x_0, by (1), we have

$$\rho(f(x_0), x_0) = \rho\Big(f(g(x_0)),\ g(x_0)\Big) \leqq \alpha\rho(x_0, g(x_0)) = 0.$$

Hence we have $f(x_0)=x_0$. Therefore x_0 is also a fixed point of f.

Let x_0, x'_0 be both fixed points of f and g, then (1) implies

$$\rho(x_0, x'_0) = \rho\Big(f(g(x_0)),\ g(x'_0)\Big) \leqq \alpha\rho(x_0, x'_0).$$

By $0<\alpha<1$, we have $\rho(x_0, x'_0)=0$, i. e. $x_0=x'_0$. We complete the proof of Theorem.

REMARK 1. Let $g(x)=x$, then we have easily seen the continuity of f, and then Theorem gives the classical theorem of S. BANACH.

REMARK 2. From the proof above, we have the following result: If there are positive numbers α, β satisfying α, $\beta < 1$ such that, for every $x \in X$,

$$\rho\big(f(g(x)),\ g(x)\big) \leq \alpha\,\rho(x, g(x)), \tag{1'}$$

$$\rho\big(g(f(x)),\ f(x)\big) \leq \beta\,\rho(x, f(x)), \tag{2'}$$

then f and g have at least one common fixed point.

REMARK 3. Let Φ be a family of continuous mappings on X into itself. If there are positive $\alpha_{f,g}$ satisfying $\alpha_{f,g} < 1$ $(f, g \in \Phi)$ such that, for $x, y \in X$,

$$\rho\big(f(g(x)),\ g(y)\big) \leq \alpha_{f,g}\,\rho(x, g(y))$$

for every $f, g \in \Phi$, then all mappings of Φ have a unique common fixed point.

REFERENCES

[1] R. E. DEMARR, *Common fixed points for commuting contruction mappings*, Pacific Jour. Math. **13** (1963), pp. 1139-1141.

[2] I. A. RUS, *Teoria punctului fix, II, Teoria punctului fix in analiţa functionala*, Universitatea «Babes-Bolyai» din Cluj, 1973.

[3] I. A. RUS, *On common fixed points*, Studia Universitatis «Babes-Bolyai», fasc. 1, 1973, pp. 31-33.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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QUASI-REGULAR NORMAL ENDOMORPHISMS OF A GROUP(*)

BY

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To Professor ALMEIDA COSTA, on his 70th birthday

INTRODUCTION

It is well known that the sum of two endomorphisms of a group G need not be an endomorphism of G, even if the endomorphisms are normal.

In this note, we observe that the set of all normal endomorphisms of a group G, constitutes a monoid with a zero element, relatively to the *circle composition*. This binary operation was first introduced by PERLIS in 1942 [1] and, afterwards, has been used by JACOBSON [2], BROWN and McCOY [4], ALMEIDA COSTA [6] and many others, in Theory of Rings.

Next, we define quasi-regular normal endomorphisms and we show that the set of all quasi-regular normal endomorphisms of a group, under circle composition, constitutes a group isomorphic to the group of all normall automorphisms.

Lastly, we study a certain type of endomorphisms defined by commutators and we apply the results obtained to the nilpotent groups of class at most two.

1. PRELIMINARIES

Let G be a multiplicative group and let e denote the neutral element of G.

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We recall that an endomorphism α of G is said to be a normal endomorphism of G, if one has

(1) $$\alpha(xyx^{-1}) = x\alpha(y)x^{-1}, \quad \text{for all } x, y \in G,$$

i. e., if α commutes with every inner automorphism of G.

The identity mapping ε ($\varepsilon(x) = x$, for every $x \in G$) and the zero mapping ω ($\omega(x) = e$, for every $x \in G$) are clearly normal endomorphisms. If, for some integer n, the mapping $x \to x^n$ is an endomorphism of G, then such an endomorphism is normal. The composite $\alpha\beta$ of the normal endomorphisms α and β is also a normal endomorphism.

If τ and α are, respectively, an automorphism and a normal endomorphism of G, then the endomorphism $\tau\alpha\tau^{-1}$ is normal.

In fact, from (1) it follows

(2) $$\begin{aligned}\tau\alpha\tau^{-1}(xyx^{-1}) &= \tau\alpha(\tau^{-1}(x)\tau^{-1}(y)\tau^{-1}(x^{-1})) = \\ &= \tau(\tau^{-1}(x)\alpha\tau^{-1}(y)\tau^{-1}(x^{-1})) = \\ &= x\tau\alpha\tau^{-1}(y)x^{-1},\end{aligned}$$

for all $x, y \in G$.

By $\alpha + \beta$ and $-\alpha$, one means the mappings defined, respectively, by

$$(\alpha+\beta)(x) = \alpha(x)\beta(x) \quad \text{and} \quad (-\alpha) = \alpha(x^{-1}),$$

for every $x \in G$.

By $\alpha - \beta$, one denotes the mapping $\alpha + (-\beta)$.

If α and β are endomorphisms of G, then $\alpha + \beta$ is an endomorphism, if and only if

(3) $$\alpha(y)\beta(x) = \beta(x)\alpha(y), \quad \text{for all } x, y \in G.$$

It is clear that, if $\alpha + \beta$ is an endomorphism, then $\beta + \alpha$ is an endomorphism.

Now, let us suppose that α and β are *normal* endomorphisms. Then, condition (3) may be written

$$\alpha(y)\alpha(\beta(x)y^{-1}\beta(x^{-1})) = e, \quad \text{for all } x, y \in G,$$

that is to say,

$$\begin{aligned}e &= \alpha(y\beta(x)y^{-1} \cdot \beta(x^{-1})) = \alpha(\beta(y)\beta(x)\beta(y^{-1})\beta(x^{-1})) = \\ &= \alpha\beta(yxy^{-1}x^{-1}),\end{aligned}$$

for all $x, y \in G$.

This means that $\alpha + \beta$ is an endomorphism, if and only if

$$G' \subseteq Ker(\alpha\beta),$$

where G' denotes the commutator subgroup of G.

If $\alpha + \beta$ is an endomorphism, then such an endomorphism is necessarily normal since

$$\begin{aligned}(\alpha+\beta)(xyx^{-1}) &= \alpha(xyx^{-1})\beta(xyx^{-1}) = x\alpha(y)\beta(y)x^{-1} = \\ &= x(\alpha+\beta)(y)x^{-1},\end{aligned}$$

for all $x, y \in G$.

It is also easy to see that, if α, β, $\alpha + \beta$ are endomorphisms and α and $\alpha + \beta$ are normal, then β is normal.

One shows easily [8] that, if α is an endomorphism, then the mapping $\varepsilon - \alpha$: $x \to x\alpha(x^{-1})$ is an endomorphism, if and only if α is normal. In addition, if α is normal, then the endomorphism $\varepsilon - \alpha$ is normal.

2. CIRCLE COMPOSITION

Let α and β be endomorphisms of G and let us suppose that α is normal. We are going to see that the mapping $\alpha + \beta - \alpha\beta$ is an endomorphism.

One has

$$(\alpha+\beta-\alpha\beta)(xy) = \alpha(x)\alpha(y)\beta(x)\beta(y)\alpha\beta(y^{-1})\alpha\beta(x^{-1})$$

and, on the other hand,

$$(\alpha+\beta-\alpha\beta)(x)(\alpha+\beta-\alpha\beta)(y) = \alpha(x)\beta(x)\alpha\beta(x^{-1})\alpha(y)\beta(y)\alpha\beta(y^{-1}),$$

for all $x, y \in G$.

We must prove that

$$\alpha(y)\beta(x)\beta(y)\cdot\alpha\beta(y^{-1})\alpha\beta(x^{-1})\alpha\beta(y)=\beta(x)\alpha\beta(x^{-1})\alpha(y)\beta(y),$$

for all $x, y \in G$.

Since α is normal, this is equivalent to

(4) $\alpha(y)\beta(x)\alpha\beta(x^{-1})=\beta(x)\alpha\beta(x^{-1})\alpha(y)$, for all $x, y \in G$.

Now, by the normality of α, one has

$$\begin{aligned}\alpha(y)\beta(x)\alpha\beta(x^{-1}) &= \beta(x)\cdot\beta(x^{-1})\alpha(y)\beta(x)\cdot\alpha\beta(x^{-1})=\\ &= \beta(x)\alpha\beta(x^{-1})\alpha(y)\alpha\beta(x)\alpha\beta(x^{-1})=\\ &= \beta(x)\alpha\beta(x^{-1})\alpha(y),\end{aligned}$$

proving (4).

In general, $\alpha+\beta-\alpha\beta$ is not a normal endomorphism.

However, *if both* α *and* β *are normal endomorphisms*, then $\alpha\beta$ is normal and so

$$\begin{aligned}(\alpha+\beta-\alpha\beta)(xyx^{-1}) &= \alpha(x)\alpha(y)\alpha(x^{-1})\cdot\beta(x)\beta(y)\beta(x^{-1})\cdot\\ &\quad\cdot\alpha\beta(x)\alpha\beta(y)\alpha\beta(x^{-1})=\\ &= x\alpha(y)x^{-1}\cdot x\beta(y)x^{-1}\cdot x\alpha\beta(y)x^{-1}=\\ &= x(\alpha+\beta-\alpha\beta)(y)x^{-1},\end{aligned}$$

for all $x, y \in G$.

In summary, the following holds:

THEOREM 1. *If* α *and* β *are endomorphisms of a group* G *and* α *is normal, then* $\alpha+\beta-\alpha\beta$ *is an endomorphism of* G; *if* α *and* β *are normal, then* $\alpha+\beta-\alpha\beta$ *is normal.*

Let E be the set of all normal endomorphisms of G and let us put

$$\alpha\circ\beta=\alpha+\beta-\alpha\beta, \quad \text{for all } \alpha, \beta \in E.$$

Thus, by Theorem 1, $\circ$ is a binary operation on E, called *circle composition.*

We are going to see that the operation $\circ$ is associative.

In fact,

$$(\alpha\circ(\beta\circ\gamma))(x)=\alpha(x)\beta(x)\gamma(x)\beta\gamma(x^{-1})\alpha\beta(x^{-1})\alpha\gamma(x^{-1})\alpha\beta\gamma(x),$$

$$((\alpha\circ\beta)\circ\gamma)(x)=\alpha(x)\beta(x)\alpha\beta(x^{-1})\gamma(x)\alpha\gamma(x^{-1})\beta\gamma(x^{-1})\alpha\beta\gamma(x).$$

We must prove that, for every $x \in G$,

(5) $\gamma(x)\beta\gamma(x^{-1})\alpha\beta(x^{-1})=\alpha\beta(x^{-1})\gamma(x)\alpha\gamma(x^{-1})\beta\gamma(x^{-1})\alpha\gamma(x)$.

Now, by the normality of β, one has

(6) $$\begin{aligned}&\alpha\beta(x^{-1})\gamma(x)\alpha\gamma(x^{-1})\beta\gamma(x^{-1})\alpha\gamma(x)=\\ &=\alpha\beta(x^{-1})\gamma(x)\cdot\beta(\alpha\gamma(x^{-1})\gamma(x^{-1})\alpha\gamma(x)).\end{aligned}$$

Let us observe that, if α is a normal endomorphism of G then one has

(7) $g\alpha(g^{-1})=g\alpha(g^{-1})g^{-1}\cdot g=\alpha(gg^{-1}g^{-1})g=\alpha(g^{-1})g$,

for every $g \in G$.

Consequently, by the normality of α and $\alpha\beta$, one concludes, from (6), that

$$\begin{aligned}\alpha\beta(x^{-1})\gamma(x)\alpha\gamma(x^{-1})\beta\gamma(x^{-1})\alpha\gamma(x) &= \alpha\beta(x^{-1})\gamma(x)\beta\gamma(x^{-1})=\\ &= \gamma(x)\cdot\gamma(x^{-1})\alpha\beta(x^{-1})\gamma(x)\cdot\beta\gamma(x^{-1})=\\ &= \gamma(x)\alpha\beta(\gamma(x^{-1})x^{-1}\gamma(x))\beta\gamma(x^{-1})=\\ &= \gamma(x)\alpha\beta(x^{-1})\beta\gamma(x^{-1}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(x)\beta\gamma(x^{-1})\alpha\beta(x^{-1}) &= \gamma(x)\cdot\beta\gamma(x^{-1})\alpha\beta(x^{-1})\beta\gamma(x)\cdot\beta\gamma(x^{-1})=\\ &= \gamma(x)\alpha(\beta\gamma(x^{-1})\beta(x^{-1})\beta\gamma(x))\beta\gamma(x^{-1})=\\ &= \gamma(x)\alpha(\beta(\gamma(x^{-1})x^{-1}\gamma(x)))\beta\gamma(x^{-1})=\\ &= \gamma(x)\alpha\beta(x^{-1})\beta\gamma(x^{-1}),\end{aligned}$$

which proves (5) and, hence, the associativity of $\circ$ holds.

One has obviously, for every $\alpha \in E$,

$$\alpha \circ \varepsilon = \alpha + \varepsilon - \alpha\varepsilon = \varepsilon = \varepsilon + \alpha - \varepsilon\alpha = \varepsilon \circ \alpha$$

and

$$\alpha \circ \omega = \alpha + \omega - \alpha\omega = \alpha = \omega + \alpha - \omega\alpha = \omega \circ \alpha .$$

Thus, we can state the following

THEOREM 2. *The set* E *of all normal endomorphisms of* G, *under circle composition, constitutes a monoid with a zero element, the neutral element being* ω *and the zero element being* ε.

THEOREM 3. *In the monoid* E, *the following holds:*

(i) $\alpha \circ \beta = \beta \circ \alpha$, *if and only if* $\alpha\beta = \beta\alpha$.
(ii) *If* $\varepsilon - \alpha$ *is injective, then* α *is left cancellable;*
(iii) *If* $\varepsilon - \alpha$ *is surjective, then* α *is right cancellable.*

PROOF. We shall prove only the part (i).
In fact, from

$$\alpha(x)\beta(x)\alpha\beta(x^{-1}) = \beta(x)\alpha(x)\beta\alpha(x^{-1}), \quad \text{for every } x \in G,$$

it results, by the normality of β,

$$\begin{aligned}\beta(x)\alpha\beta(x^{-1}) &= \alpha(x^{-1})\beta(x)\alpha(x)\cdot\beta\alpha(x^{-1}) = \\ &= \beta\alpha(x^{-1})\beta(x)\beta\alpha(x)\cdot\beta\alpha(x^{-1}) = \\ &= \beta\alpha(x^{-1})\beta(x).\end{aligned}$$

By the normality of α, one has, according to (7),

$$\beta(x)\alpha\beta(x^{-1}) = \beta(\alpha(x^{-1})x) = \beta(x\alpha(x^{-1})) = \beta(x)\beta\alpha(x^{-1}),$$

for every $x \in G$.
This implies obviously $\alpha\beta = \beta\alpha$.
Conversely, if one has $\alpha\beta = \beta\alpha$, then, in order to conclude that $\alpha \circ \beta = \beta \circ \alpha$, it suffices clearly to show that

$$\alpha(x)\beta(x) = \beta(x)\alpha(x), \quad \text{for every } x \in G.$$

Now, by the normality of α and β and by (7), one has, for every $x \in G$,

$$\begin{aligned}\alpha(x)\beta(x) &= \alpha(x)\beta(x)\alpha(x^{-1})\cdot\alpha(x) = \\ &= \beta\alpha(x)\beta(x)\beta\alpha(x^{-1})\alpha(x) = \\ &= \beta(\alpha(x)x\alpha(x^{-1}))\alpha(x) = \\ &= \beta(x)\alpha(x),\end{aligned}$$

as wanted.

3. QUASI-REGULARITY

A normal endomorphism α is said to be *right* (resp. *left*) *quasi-regular*, if there exists some normal endomorphism α' (resp. α'') such that

$$\alpha + \alpha' - \alpha\alpha' = \omega \qquad (\text{resp. } \alpha'' + \alpha - \alpha''\alpha = \omega).$$

Such an element α' (resp. α'') is called a *right* (resp. *left*) *quasi-inverse* of α.

One says that α is *quasi-regular*, if α is right and left quasi-regular. If α is quasi-regular and α' and α'' are, respectively, a right quasi-inverse of α and a left quasi-inverse of α, then it is immediate that $\alpha' = \alpha''$ and one says that α' is *the quasi-inverse* of α.

It is clear that the normal endomorphism ε is neither right quasi-regular nor left quasi-regular and that the normal endomorphism ω is quasi-regular, being $\omega' = \omega$.

The set Q of all quasi-regular normal endomorphisms of G is a sub-group of the monoid E (Q is the set of units of the monoid E).

Let α be a quasi-regular normal endomorphism of G. The condition

$$\alpha + \alpha' - \alpha\alpha' = \omega .$$

is equivalent to

$$x\alpha(x)\alpha'(x)\alpha\alpha'(x^{-1}) = x, \quad \text{for every } x \in G.$$

Since α and α' are normal, this condition is equivalent to

$$x = (\varepsilon - \alpha)(x)(\varepsilon - \alpha)(\alpha'(x^{-1})) = (\varepsilon - \alpha)(x\alpha'(x^{-1})) = (\varepsilon - \alpha)(\varepsilon - \alpha')(x),$$

for every $x \in G$.

By a similar way, one sees that

$$x=(\varepsilon-\alpha')(\varepsilon-\alpha)(x), \quad \text{for every } x\in G.$$

In other words, one has

$$(\varepsilon-\alpha)(\varepsilon-\alpha')=\varepsilon=(\varepsilon-\alpha')(\varepsilon-\alpha),$$

and from this it follows

$$\alpha'=\varepsilon-(\varepsilon-\alpha')=\varepsilon-(\varepsilon-\alpha)^{-1}.$$

In short, the following holds:

THEOREM 4. *The endomorphism* α *of the group* G *is a quasi--regular normal endomorphism, if and only if* $\varepsilon-\alpha$ *is an automorphism; if* α *is a quasi-regular normal endomorphism, then the quasi-inverse of* α *is* $\varepsilon-(\varepsilon-\alpha)^{-1}$.

The nilpotent normal endomorphisms are seen to be quasi-regular. Thus, if $\alpha^{n+1}=\omega$ for some positive integer n, then one shows that one has

$$\alpha'=-(\alpha+\alpha^2+\cdots+\alpha^n).$$

Now, let us suppose that σ is an automorphism of G. Then, σ is normal, if and only if

$$\sigma(x^{-1})\sigma(y)\sigma(x)=x^{-1}\sigma(y)x, \quad \text{for all } x,y\in G,$$

which is obviously equivalent to

$$x\sigma(x^{-1})\cdot\sigma(y)=\sigma(y)\cdot x\sigma(x^{-1}), \quad \text{for all } x,y\,G.$$

Since σ is onto, one concludes that σ is normal, if an only if $x\sigma(x^{-1})$ belongs to the center Z of G.

Let A be the set of normal automorphisms of G.

It is clear that, if $\sigma,\tau\in A$, then $\sigma\tau\in A$. Moreover, if one has $x\sigma(x^{-1})\in Z$, for every $x\in G$, then

$$(\sigma^{-1}(x)x^{-1})^{-1}=x\sigma^{-1}(x^{-1})\in Z, \quad \text{for every } x\in G.$$

because

$$\sigma^{-1}(x)x^{-1}=\sigma^{-1}(x\sigma(x^{-1}))$$

and the center is a characteristic subgroup.

Hence, it follows $\sigma^{-1}\in A$. Consequently, A is a subgroup of the group of the automorphisms of G.

Let us consider the mapping $f\colon Q\to A$, defined by $f(\alpha)=\varepsilon-\alpha$, for every $\alpha\in Q$. It is immediate that f is a bijective mapping and, in addition, for all $\alpha,\beta\in Q$, one has

$$f(\alpha\circ\beta)=\varepsilon-(\alpha+\beta-\alpha\beta)=(\varepsilon-\alpha)(\varepsilon-\beta)=f(\alpha)f(\beta).$$

Thus, we can state the following

THEOREM 5. *The set of all normal automorphisms of* G *is a subgroup of the group of all automorphisms of* G, *isomorphic to the group of all quasi-regular normal endomorphisms of* G.

Now let us suppose that σ is a quasi-regular normal automorphism of G. Then, one has, for every $x\in G$,

$$\sigma(x)\sigma'(x)\sigma\sigma'(x^{-1})=e$$

and, in addition,

$$\sigma'(x)\sigma(\sigma'(x^{-1}))\in Z.$$

From this it follows that $\sigma(x)\in Z$, for every $x\in Z$.

This means that $Z=G$ and so the following holds:

THEOREM 6. *If there exists some quasi-regular normal automorphism of* G, *then* G *is an Abelian group.*

4 — ENDOMORPHISMS DEFINED BY COMMUTATORS

Let c be an element of G such that, for every $x\in G$, the commutator $[c,x]=c^{-1}x^{-1}cx$ belongs to the center Z of G and let us consider the mapping $\Gamma_c\colon G\to G$, defined by

$$\Gamma_c(x)=[c,x], \quad \text{for every } x\in G. \tag{8}$$

One has $\Gamma_c(xy)=[c,xy]=c^{-1}y^{-1}x^{-1}cxy$ and, on the other hand,

$$\Gamma_c(x)\Gamma_c(y)=c^{-1}x^{-1}cx\cdot c^{-1}y^{-1}cy=c^{-1}y^{-1}c\cdot c^{-1}x^{-1}cx\cdot y=$$
$$=c^{-1}y^{-1}x^{-1}cxy=\Gamma_c(xy),$$

because $c^{-1}x^{-1}cx\in Z$.

This means that Γ_c is an endomorphism and, since $\Gamma_c(x)\in Z$, for every $x\in G$, this endomorphism is normal.

It is easy to see that this endomorphism belongs to the center of the monoid E. In fact, by Theorem 2, (*i*), it suffices to show that

$$(9)\qquad \alpha\Gamma_c=\Gamma_c\alpha, \quad \text{for every} \quad \alpha\in E.$$

One has clearly, for every $x\in G$,

$$(\alpha\Gamma_c)(x)=\alpha(c^{-1}x^{-1}cx)=c^{-1}\alpha(x^{-1})c\,\alpha(x)=[c,\alpha(x)]=$$
$$=\Gamma_c(\alpha(x))=(\Gamma_c\alpha)(x),$$

which proves (9).

Let us observe that $c^{-1}x^{-1}cx=z\in Z$ implies

$$[c^{-1},x]=cx^{-1}c^{-1}x=c(x^{-1}cx)^{-1}=c(cz)^{-1}=z^{-1}\in Z.$$

We are going to see that the mapping $\Gamma'_c:\ x\to[c^{-1},x]=\Gamma_{c^{-1}}(x)$, which is clearly a normal endomorphism, is the quasi-inverse of Γ_c (it suffices obviously to show that Γ'_c is a right quasi-inverse of Γ_c). Let us note that, if Γ_c and Γ_d are endomorphisms defined by commutators, as in (8), then

$$\Gamma_c\Gamma_d=\omega,$$

since $\Gamma_c\Gamma_d(x)=[c,[d,x]]=e$, because $[d,x]\in Z$, for every $x\in G$.

This means that $\Gamma_c\circ\Gamma_d=\Gamma_c+\Gamma_d$.

Consequently, one has, for every $x\in G$,

$$(\Gamma_c+\Gamma'_c-\Gamma_c\Gamma'_c)(x)=\Gamma_c(x)\Gamma'_c(x)=c^{-1}x^{-1}cx\cdot cx^{-1}c^{-1}x=$$
$$=c\cdot c^{-1}x^{-1}cx\cdot x^{-1}c^{-1}x=e,$$

as wanted.

Since

$$(\Gamma_c\circ\Gamma_d)(x)=\Gamma_c(x)\Gamma_d(x)=c^{-1}x^{-1}cx\cdot d^{-1}x^{-1}dx=$$
$$=c^{-1}\cdot d^{-1}x^{-1}dx\cdot x^{-1}cx=c^{-1}d^{-1}x^{-1}dcx,$$

one has, by taking into account the fact that $d^{-1}c^{-1}dc\in Z$,

$$(\Gamma_c\circ\Gamma_d)(x)=d^{-1}c^{-1}dc\cdot c^{-1}d^{-1}x^{-1}dcx\cdot c^{-1}d^{-1}cd=$$
$$=d^{-1}c^{-1}x^{-1}dc\cdot c^{-1}d^{-1}cdx=d^{-1}c^{-1}x^{-1}cdx=$$
$$=[cd,x]=[c,x][d,x]\in Z.$$

In short, the following holds:

THEOREM 7. *The set* C *of all endomorphisms of* G *defined by commutators as in* (8), *is a normal subgroup of the group* Q, *constituted by all quasi-regular normal endomorphisms of* G, *and it is contained in the center of the monoid* E, *formed by the normal endomorphisms of* G.

THEOREM 8. *If* Γ_c *is the endomorphism of* G *defined by* (8), *and* τ *is an automorphism of* G, *then* $\tau\Gamma_c\tau^{-1}=\Gamma_{\tau(c)}$.

PROOF. Indeed, by (2), one knows that $\tau\Gamma_c\tau^{-1}$ is a normal endomorphism.

In addition, one has, for every $x\in G$,

$$\tau\Gamma_c\tau^{-1}(x)=\tau(c^{-1}\tau^{-1}(x^{-1})c\,\tau^{-1}(x))=\tau(c^{-1})x^{-1}\tau(c)\,x=[\tau(c),x]\in Z,$$

proving theorem above.

5. NILPOTENT GROUPS OF CLASS AT MOST TWO

Let c be an element of G and let us consider the inner automorphism of G, $\theta_c: x\to cxc^{-1}$. It is immediate that θ_c is normal, if and only if the commutator $[c,x]$ is in the center Z of G, for every $x\in G$, i.e., θ_c is normal, if and only if the mapping Γ_c is an endomorphism of G.

Let θ_c be a normal endomorphism and let $\theta_c = \theta_d$; then one has clearly $c = dz$ for some $z \in Z$ and, consequently, $[c, x] = [d, x]$, for every $x \in G$.

Conversely, if $[c, x] = [d, x]$, for every $x \in G$, then one has obviously $\theta_c = \theta_d$.

Thus, if I denotes the group of the inner automorphisms of G, by setting $h(\theta_c) = \Gamma_c$, one gets a bijective mapping $h\colon A \cap I \to C$.

Since

$$h(\theta_c \theta_d) = h(\theta_{cd}) = \Gamma_{cd}$$

and, for every $x \in G$,

$$\Gamma_{cd}(x) = [cd, x] = [c, x][d, x] = \Gamma_c(x)\,\Gamma_d(x)\,\Gamma_c\,\Gamma_d(x^{-1}) = $$
$$= (\Gamma_c \circ \Gamma_d)(x),$$

one has

$$h(\theta_c \theta_d) = \Gamma_c \circ \Gamma_d = h(\theta_c) \circ h(\theta_d),$$

and, therefore, the following holds:

THEOREM 9. *For every group* G, *the corresponding group* C *is isomorphic to the group of all normal inner automorphisms of* G.

Now, let us suppose that G is a nilpotent group of class at most two, that is to say, $G' \subseteq Z$.

Then $A \cap I = I$, in view of the fact that, under the hypothesis above, every inner automorphism is normal.

Conversely, if $A \cap I$, then one has clearly $G' \subseteq Z$.

Consequently, one can state the following

THEOREM 10. *A group* G *is nilpotent of class at most two, if and only if the mapping* $h\colon I \to C$, *defined by* $h(\theta_c) = \Gamma_c$, *is an isomorphism.*

BIBLIOGRAPHY

[1] S. PERLIS, *A characterization of the radical of an algebra*, Bull. Am. Math. Soc. **48** (1942), pp. 128-132.

[2] N. JACOBSON, *The radical and semi-simplicity for arbitrary rings*, Am. J. Math., **48** (1947), pp. 8-21.

[3] ———, *Lectures in Abstract Algebra*, vol. I, New York, 1951.

[4] B. BROWN and N. McCOY, *The radical of a ring*. Duke Math. J., **15** (1948), ps. 495-499.

[5] N. McCOY, *The Theory of Rings*, second printing, New York, 1965.

[6] A. ALMEIDA COSTA, *Sobre nilideais e ideais quase-regulares*, Anais Fac. Ci. Porto, **34** (1950), pp. 65-74 e 129-144.

[7] ———, *Cours d'Algèbre Générale*, vol. I, Lisboa, 1964; vol. II, Lisboa, 1968.

[8] J. MORGADO, *A note on the normal endomorphisms of a group*, Gaz. Mat. **109-112** (1968), pp. 6-8.

[9] N. DIVINSKY, *Rings and Radicals*, Toronto, 1965.

[10] A. KUROSH, *The Theory of Groups*, vol. I and vol. II, New York, 1960.

[11] I. MACDONALD, *The Theory of Groups*, Oxford, 1968.

[12] J. ROTMAN, *The Theory of Groups: An Introduction*, Boston, 1966.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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ON A FAITHFUL REPRESENTATION OF SIMILARLY DECOMPOSABLE SEMIGROUPS (*)

BY

OTTÓ STEINFELD
Budapest, Hungary

Dedicated to Professor A. ALMEIDA COSTA

1. Let S be a semigroup with the zero element 0. The left ideals L_1 and L_2 of S are said to be *left similar* if there exists a one-to-one mapping φ of L_1 onto L_2 such that

(1.1) $$(sx)\varphi = s(x\varphi) \quad \text{for all} \quad x \in L_1 \quad \text{and} \quad s \in S.$$

(See our paper [2]).

Right similarity of the right ideals R_1 and R_2 of S is defined dually.

Let S be a semigroup with 0 such that

(1.2) $$S = \bigcup_{\lambda \in \Lambda} Se_\lambda = \bigcup_{i \in I} e_i S \qquad (e_\lambda^2 = e_\lambda;\ e_i^2 = e_i;\ 1 \in I \cap \Lambda)$$

where $Se_\lambda (\lambda \in \Lambda) [e_i S (i \in I)]$ are left [right] similar left [right] ideals of S such that $Se_\mu \cap Se_\nu = 0$ $(\mu, \nu \in \Lambda;\ \mu \neq \nu)$ and $e_j S \cap e_k S = 0$ $(j, k \in I;\ j \neq k)$. We call a semigroup S with these properties *similarly decomposable.*

The following characterization of completely 0-simple semigroups is proved in our paper [2].

PROPOSITION 1.1. *A semigroup* S *with* 0 *is completely* 0-*simple if and only if* S *has the form*

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(1. 3) $$S = \bigcup_{\lambda \in \Lambda} S e_\lambda \qquad (e_\lambda^2 = e_\lambda)$$

where the $S e_\lambda$ *are pairwise left similar 0-minimal left ideals of* S.

From Proposition 1. 1 and its dual we obtain that the completely 0-simple semigroups are similarly decomposable.

The purpose of this paper is to give a faithful representation of similarly decomposable semigroups by special (infinite) matrices.

A matrix $A = (a_{ij})$ $(i \in X;\ j \in Y;\ a_{ij} \in H)$ over a semigroup H with 0 is called *row-monomial* if each row of A contains at most one non-zero element of H. If every non-zero element of A lies in the same column then we say that A is a *row-monomial matrix with a unique non-zero column.*

Dually one can define a *column-monomial matrix with a unique non-zero row.*

It is easy to show that the set of all row-monomial $X \times X$ matrices $A = (a_{ij})$ $(i, j \in X;\ a_{i\cdot} \in H)$ with a unique non-zero column over H is a semigroup concerning the usual multiplication of matrices over a semigroup. (See A. H. Clifford and G. B. Preston [1] section 3. 1).

2. First we need the following known result:

Proposition 2. 1. (See O. Steinfeld [2] Proposition 2. 1). *Let* S *be a semigroup with* 0 *and* $e_1 \neq 0$, $e_2 \neq 0$ *idempotents in* S. *Then the left ideals* $S e_1$ *and* $S e_2$ *are left similar if and only if there exist elements* q_{12} *and* q_{21} *in* S *such that*

(2. 1) $$e_1 q_{12} e_2 = q_{12}, \qquad e_2 q_{21} e_1 = q_{21},$$

(2. 2) $$q_{12} q_{21} = e_1, \qquad q_{21} q_{12} = e_2.$$

Naturally, Proposition 2. 1 has a dual concerning the right ideals $e_1 S$ and $e_2 S$ of S.

Now the preliminaries are ready for

Theorem 2. 2. *Let* S *be a similarly decomposable semigroup having the form*

(2. 3) $$S = \bigcup_{\lambda \in \Lambda} S e_\lambda = \bigcup_{i \in I} e_i S \qquad (1 \in I \cap \Lambda).$$

Then there exist elements $q_{1\lambda} (\in e_1 S e_\lambda)$, $q_{\lambda 1} (\in e_\lambda S e_1)$ $(\lambda \in \Lambda)$ *and* $r_{1i} (\in e_1 S e_i)$, $r_{i1} (\in e_i S e_1)$ $(i \in I)$ *such that*

(2. 4) $$q_{1\lambda} q_{\lambda 1} = e_1, \quad q_{\lambda 1} q_{1\lambda} = e_\lambda \qquad (\lambda \in \Lambda),$$

(2. 5) $$r_{1i} r_{i1} = e_1, \qquad r_{i1} r_{1i} = e_i \qquad (i \in I),$$

and

(2. 6) $$\Phi:\ a = e_j a = a e_\mu = e_j a e_\mu \to \Phi(a) =$$

$$= \begin{array}{c} \\ \\ \lambda \\ \\ \end{array} \begin{array}{c} \quad \mu \quad \\ \begin{bmatrix} 0 \cdots 0 & q_{11} a q_{\mu 1} & 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots 0 & q_{1\lambda} a q_{\mu 1} & 0 \cdots 0 \\ \vdots & \vdots & \vdots \end{bmatrix} \end{array} \qquad (a \in S;\ j \in I;\ \lambda, \mu \in \Lambda)$$

is a homomorphic mapping of S *into the semigroup of all row-monomial* $\Lambda \times \Lambda$ *matrices with a unique non-zero column over the semigroup* $e_1 S e_1$, *furthermore*

(2. 7) $$\Psi:\ a = e_j a = a e_\mu = e_j a e_\mu \to \Psi(a) =$$

$$= \begin{array}{c} \\ \\ \\ \\ j \\ \\ \\ \\ \end{array} \begin{array}{c} \quad\quad\quad i \quad\quad\quad \\ \begin{bmatrix} 0 & \cdots & 0 & \cdots \\ \vdots & \cdots & \vdots & \cdots \\ 0 & \cdots & 0 & \cdots \\ r_{1j} a r_{11} & \cdots & r_{1j} a r_{i1} & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots \end{bmatrix} \end{array} \qquad (a \in S;\ i, j \in I;\ \mu \in \Lambda)$$

is a homomorphic mapping of S *into the semigroup of all column-monomial* $I \times I$ *matrices with a unique non-zero row over the semigroup* $e_1 S e_1$. *Finally, if for the elements* $a, b \in S$ *and for the mappings* (2. 6) *and* (2. 7) *the equations*

(2. 8) $$\Phi(a) = \Phi(b) \quad and \quad \Psi(a) = \Psi(b)$$

hold, then $a = b$.

PROOF. By proposition 2. 1 and its dual there exist elements $q_{1\lambda}, q_{\lambda 1}$ and r_{1i}, r_{i1} with the properties (2. 4), (2. 5), indeed.

Let $a = e_j a e_\mu (j \in I, \mu \in \Lambda)$ and $c = e_k c e_\nu (k \in I, \nu \in \Lambda)$ be any two elements of S, then by (2. 6) and (2. 4)

$$\Phi(a)\cdot\Phi(c) = \begin{array}{c} \\ \\ \lambda \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & \mu & \end{array} \\ \begin{bmatrix} 0\dots 0 & q_{11}\, a\, q_{\mu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \\ 0\dots 0 & q_{1\lambda}\, a\, q_{\mu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \end{bmatrix}\end{array} \cdot \begin{array}{c} \begin{array}{ccc} & \nu & \end{array} \\ \begin{bmatrix} 0\dots 0 & q_{11}\, c\, q_{\mu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \\ 0\dots 0 & q_{1\lambda}\, c\, q_{\mu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \end{bmatrix}\end{array} =$$

$$= \begin{array}{c} \\ \\ \lambda \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & \nu & \end{array} \\ \begin{bmatrix} 0\dots 0 & q_{11}\, a e_\mu c\, q_{\nu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \\ 0\dots 0 & q_{1\lambda}\, a e_\mu c\, q_{\nu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \end{bmatrix}\end{array}.$$

On the other hand, again by (2. 6),

$$\Phi(ac) = \Phi(e_j a c e_\nu) = \begin{array}{c} \\ \\ \lambda \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} & \nu & \end{array} \\ \begin{bmatrix} 0\dots 0 & q_{11}\, a c\, q_{\nu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \\ 0\dots 0 & q_{1\nu}\, a c\, q_{\nu 1} & 0\dots 0 \\ \vdots & \vdots & \vdots \end{bmatrix}\end{array}.$$

Since $a = a e_\mu$, we have $\Phi(a)\cdot\Phi(c) = \Phi(ac)$, that is, Φ is a homomorphic mapping, indeed. As all $q_{1\varrho}\, x\, q_{\sigma 1} (x \in S; \varrho, \sigma \in \Lambda)$ are elements of the subsemigroup $e_1 S e_1$ of S, our assertions concerning Φ are true.

Similarly one can prove the assertions concerning the mapping Ψ.

Finally assume that for the elements $a = e_j a e_\mu$ and $b = e_{j'} b e_{\mu'}$ of S and for the mappings Φ and Ψ

$$\Phi(a) = \Phi(b) \quad \text{and} \quad \Psi(a) = \Psi(b)$$

hold, then (2. 6) and (2. 7) imply $j = j'$ and $\mu = \mu'$ furthermore

$$q_{1\lambda}\, a\, q_{\mu 1} = q_{1\lambda}\, b\, q_{\mu 1} \qquad (\lambda \in \Lambda), \tag{2. 9}$$

$$r_{1j}\, a\, r_{i1} = r_{1j}\, b\, r_{i1} \qquad (i \in I) \tag{2. 10}$$

must hold.

From (2. 9), (2. 4) and (2. 6) we obtain

$$\begin{aligned} q_{1\lambda}\, a &= q_{1\lambda}\, a\, e_\mu = q_{1\lambda}\, a\, q_{\mu 1}\cdot q_{1\mu} = q_{1\lambda}\, b\, q_{\mu 1}\cdot q_{1\mu} = \\ &= q_{1\lambda}\, b\, e_\mu = q_{1\lambda}\, b \qquad (\lambda \in \Lambda). \end{aligned} \tag{2. 11}$$

Consider the idempotent element $e_j (j \in I)$. By the decomposition (1. 2) it is contained it is contained in a left ideal $S e_\varrho (\varrho \in \Lambda)$. Hence $e_j = e_j e_\varrho$. This, (2. 4), (2. 6) and (2. 11) imply

$$a = e_j a = e_j e_\varrho a = e_j q_{\varrho 1} q_{1\varrho} a = e_j q_{\varrho 1} q_{1\varrho} b = e_j e_\varrho b = e_j b = b.$$

q. e. d.

COROLLARY. *Let* S *be a similarly decomposable semigroup. Then, using the notations of the previous theorem,*

$$\Gamma:\ \mathrm{a} = \mathrm{e_j\, a} = \mathrm{a\, e_\mu} = \mathrm{e_j\, a\, e_\mu} \to \Gamma(\mathrm{a}) =$$

$$= \begin{array}{c} \left.\begin{array}{c} \\ \lambda \\ \\ \end{array}\right\}\Lambda \\ \\ \left.\begin{array}{c} \\ \\ \\ \mathrm{j} \\ \\ \\ \end{array}\right\}\mathrm{I} \end{array} \begin{array}{c} \begin{array}{cc} \mu & \end{array} \\ \left[\begin{array}{ccc|cccc} 0\dots 0 & \mathrm{q_{11}\, a\, q_{\mu 1}} & 0\dots & & & & \\ \vdots & \vdots & \vdots & & 0 & & \\ 0\dots 0 & \mathrm{q_{1\lambda}\, a\, q_{\mu 1}} & 0\dots & & & & \\ \vdots & \vdots & \vdots & & & & \\ \hline & & & 0 & \dots & 0 & \dots \\ & & & \vdots & \dots & \vdots & \dots \\ & & & 0 & \dots & 0 & \dots \\ & 0 & & \mathrm{r_{1j}\, a\, r_{11}} & \dots & \mathrm{r_{1j}\, a\, r_{i1}} & \dots \\ & & & 0 & \dots & 0 & \dots \\ & & & \vdots & \dots & \vdots & \dots \end{array}\right] \\ \begin{array}{cc} \underbrace{\qquad\qquad\qquad}_{\Lambda} & \underbrace{\qquad\qquad\qquad}_{\mathrm{I}\quad \mathrm{i}} \end{array} \end{array}$$

is a faithful representation of the semigroup S.

This Corollary was noticed by my colleague L. MÁRKI.

3. Remark 1. Consider again the similarly decomposable semigroup S having the form (2.3) and consider elements $q_{1\lambda}(\in e_1 S e_\lambda)$, $q_{\lambda 1}(\in e_\lambda S e_1)$ $(\lambda \in \Lambda)$ and $r_{1i}(\in e_1 S e_i)$, $r_{i1}(\in e_i S e_1)$ $(i \in I)$ with properties (2.4) and (2.5). Furthermore, let $M^0 = M^0(e_1 S e_1;\ I,\ \Lambda;\ P)$ denote the Rees matrix semigroup over the semigroup $e_1 S e_1$ with the sandwich matrix $P = (q_{1\lambda}\, r_{i1})$.

Generalizing the well-known Rees Theorem of completely 0-simple semigroups, we have proved that the mapping

$$\varphi:\ a = e_j\, a\, e_\mu \to (r_{1j}\, a\, q_{\mu 1})_{j\mu} \qquad (a \in S;\ j \in I,\ \mu \in \Lambda) \tag{3.1}$$

is an isomorphism of S onto M^0.

It is not difficult to show that

$$\Phi(a) = P \cdot (r_{1j}\, a\, q_{\mu 1})_{j\mu} \text{ and } \Psi(a) = (r_{1j}\, a\, q_{\mu 1})_{j\mu} \cdot P \tag{3.2}$$

where Φ and Ψ denote the mappings determined by (2.6) and (2.7), respectively.

Remark 2. As every completely 0-simple semigroup is similarly decomposable, Theorem 2.2 and its Corollary yield a faithful representation of completely 0-simple semigroups. Thus the relations (3.2) hold for these semigroups, too. (Cf. Theorem 3.17 in the book [1] of A. H. Clifford and G. B. Preston).

REFERENCES

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups, I.* (Providence, 1961).

[2] O. Steinfeld, *On a generalization of completely 0-simple semigroups*, Acta Sci. Math. **28** (1967), 135-145.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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ON THE TORSION UNITS OF FINITE GROUP RINGS(*)

BY

HANS ZASSENHAUS

Devoted to ALMEIDA COSTA

We know that the order of any unit of finite order («torsion unit») of the group ring, $\mathbf{Z}[G]$, of a finite group, G, over the rational integer ring, $\mathbf{Z}$, divides the product of 2 by the order of the group, $|G|$. This result is still far away from the conjectured result that any torsion unit of $\mathbf{Z}[G]$ is conjugate under the inner automorphisms of the group ring, $\mathbf{Q}[G]$, of G over the rational number field, $\mathbf{Q}$, to some group element or its negative.

Torsion units of the group ring, $I[G]$, of G over an arbitrary integral domain, I, are trivially obtained in the form ζg with ζ any torsion element of the unit group, $U(I)$, of I.

Recently Sudarshan K. Sehgal asked the question under which conditions on the integral domain I it is always true that the order of a torsion element of the factor group of the unit group, $U(I[G])$, of $I[G]$ over the central subgroup $U(I)1_G$ is a divisor of the order of G.

If I is of prime characteristic then already the group ring of G over the subring, I_0, of I that is generated by 1_I provides us with counter examples unless G consists of only one element.

If I is of zero characteristic so that the rational number field is the prime field of the quotient field, F, of I then there are counter examples for which

$$I \cap |G|^{-1}\mathbf{Z} = d^{-1}\mathbf{Z} \tag{1}$$

(*) Recebido em 7 de Fevereiro de 1974.

for any given natural number $d>1$ dividing $|G|$ such that d and $|G|/d$ are coprime.

Indeed, let G a cyclic group of order $n>1$. Let d a natural number dividing n such that $d>1$ and $d, n/d$ are coprime. Let I_0 be the subring of $\mathbf{Q}$ formed by all rational numbers with denominator prime to n/d. Then the integral closure, I, of I_0 in the cyclotomic field $\mathbf{Q}(\zeta)$ generated by the adjunction of a primitive m-th unit root ζ to $\mathbf{Q}$ satisfies (1).

But the idempotent $e = n^{-1}\Sigma_{g\in G}\, g^{n/d}$ is contained in $I[G]$ such that the element $\zeta e + 1 - e$ is a torsion unit of $I[G]$ of order m equal to the order of its residue class modulo $U(I)1_G$ in the factor group $U(I[G]/U(I)1_G$. Choosing m as a natural number not dividing n we obtain a counter example.

Theorem. *Let* G *a finite group. Let the integral domain* I *of the field* F *satisfy the condition*

$$\mathrm{I} \cap |\mathrm{G}|^{-1}\mathbf{Z} = \mathbf{Z},$$

in other words no prime divisor of the group order is a unit. Then the order of any torsion element of $\mathrm{U(I[G])/U(I)1_G}$ *is a divisor of the exponent of* G. *Moreover, if the order is greater than* 1 *then the regular trace of any representative is zero.*

Proof. Without loss of generality assume that I is integrally closed in the algebraically closed field F (replace I by its integral closure in the algebraic hull of G!). We note that for any natural number n and for any element α of I there is always a solution of the equation $\xi^n = \alpha$ in I.

Now we have to show that for any element

$$a = \Sigma_{g\in G}\lambda(g)g \quad (\lambda:\ G \to I) \tag{3}$$

of finite order n in $U(I[G])$ such that

$$a^j \notin U(I)1_G \quad (1 \leq j < n) \tag{4}$$

always n divides the exponent of G and that

$$n > 1 \Rightarrow \lambda(1_G) = 0. \tag{5}$$

For this purpose it suffices to prove

Lemma 1. *If* $\lambda(1) \neq 0$ *then* $\mathrm{n} = 1$.

Lemma 2. *If the regular trace defined for* I[G] *over* I *is zero on all powers of* a *excepting the* n-th *power* 1_G *then* n *divides the order of* G.

Lemma 3. *If the order of* a *is a power of a prime number* p, *say* p^ν, *then*

$$(\Sigma_{g\in G}\,\lambda(g))^{p^\nu} \equiv 1 \,(\mathrm{mod}\ \mathrm{p\,I}) \tag{6}$$

$$|g| = p^\nu.$$

Hence p^ν divides the exponent of the group as a consequence of (2) and Lemma 1, 2.

In order to prove Lemma 1 we need

Lemma 4. *Given an integral domain* $\mathrm{I_0}$ *contained in the field* F, *a quadratic matrix* X *of degree* m *over* F *and* n *matrices* $\mathrm{M_1, M_2, \cdots, M_n}$ *of* $\mathrm{F^{q\times m}}$ *such that the composite matrix* $\mathrm{M} = (\mathrm{M_1, M_2, \cdots, M_n})$ *satisfies the condition*

$$(\mathrm{M_1X, M_2X, \cdots, M_nX}) \in \mathrm{M\,I_0^{nm\times nm}}$$

and that at least one of the matrices $\mathrm{M_i}$ *has maximal rank* m *then the coefficients of* X *belong to the integral closure of* $\mathrm{I_0}$ *in* F.

This Lemma is of intrinsic interest inasmuch as it generalizes the well known Kronecker criterion for elements ξ of F to be integral over I_0 precisely if there is a non zero finitely generated I_0-submodule of F invariant under multiplication by ξ.

Proof of Lemma 4:

1. If X is a diagonal matrix then Lemma 4 follows by application of Kronecker's criterion. Note that because of the maximality of the rank of at least one of the M_j's that matrix has no zero column.

2. If there are matrices P, Q of $GL(m, I_0)$ such that PXQ is diagonal then there holds the relation

$$M(P^{-1}\delta_{\alpha\beta})(PXQ\delta_{\alpha\beta}) \in MI_0^{nm\times nm}(Q\delta_{\alpha\beta})$$
$$=M(P^{-1}\delta_{\alpha\beta})I_0^{nm\times nm} \qquad (\alpha, \beta=1, 2, \cdots, n)$$

so that the coefficients of PXQ belong to the integral closure of I_0 in F in accordance with 1. The same is true for the coefficients of $X=P^{-1}(PXQ)Q^{-1}$.

3. If I_0 is a Prüfer domain then there are always matrices P, Q of $GL(m, I_0)$ for which PXQ is diagonal. Hence Lemma 4 holds for Prüfer domains.

Krull domains are Prüfer domains so that Lemma 4 also holds for Krull domains.

4. The intersection of all Krull domains of F with F as quotient field and I_0 as subring is the integral closure of I_0 in F. Thus Lemma 4 is established in accordance with 3.

We apply Lemma 4 to $I_0=I\cap\mathbf{Q}$ and to the matrix

$$X=(\Sigma_{g\in G}\lambda(g)\delta_{ig,k}) \qquad (i, k\in G)$$

giving the right regular representation of a with respect to the group elements as I-basis of $I[G]$ over I with $m=|G|$ and to the matrices

$$M_1=I_{|G|}, \; M_2=X, \cdots, M_n=X^{n-1}$$

each of rank $|G|$ in $F^{|G|\times|G|}$.

The equation $a^n=1$ implies the matrix equation $X^n=I_{|G|}$ which implies the relation

$$M(X\delta_{\alpha\beta}) \in MI_0^{n|G|\times n|G|} \qquad (\alpha, \beta=1, 2, \cdots, n)$$

for $M=(M_1, M_2, \cdots, M_n)$.

It follows that all coefficients of X belong to the integral closure of I_0 in F. In particular $\lambda(1_G)$ belongs to it.

Proof of Lemma 1: The regular trace of a is equal to $\lambda(1_G)|G|$. The equation $a^n=1_G$ implies that

$$\lambda(1_G)|G|=\Sigma_{j=1}^{|G|}\xi_j \tag{7}$$

where the ξ_j's are elements of I satisfying the equation $\xi_j^n=1$. Hence the right hand side of (7) is an algebraic integer. By our construction the absolute value of $\lambda(1_G)$ is not greater than 1. The same applies to each conjugate of $\lambda(1_G)$ under the automorphism group of F.

We note that the number of distinct conjugates of the algebraic number $\lambda(1_a)$ under the automorphism group of F is a natural number μ and that the μ conjugates form a full set of algebraic conjugates of $\lambda(1_G)$ over $\mathbf{Q}$, due to the algebraic closure property of F. The μ algebraic conjugates of $\lambda(1_G)$ belong to the integral closure of I_0 in F where I_0 is an integrally closed subring of $\mathbf{Q}$.

Hence their product is a non zero element of I_0 of absolute value not greater than 1. But by construction it belongs to $I\cap|G|^{-\mu}\mathbf{Z}$ which is $\mathbf{Z}$ by (2). Hence it is ± 1 and

$$\xi_j=\xi_1 \qquad (1\leqq j\leqq|G|)$$
$$a=\xi_1 1_G$$
$$\xi_1=1$$
$$a=1_G$$
$$n=1$$

q. e. d.

Proof of Lemma 2: The element

$$e=n^{-1}\Sigma_{j=1}^n a^j$$

of $F[G]$ has regular trace $n^{-1}|G|$. On the other hand it is idempotent by construction so that its regular trace is a rational integer q. e. d.

Proof of Lemma 3: E. Landau observed with regard to the development of the m-th power of a finite sum $x_1+x_2+\cdots+x_r$ in a given ring R as a sum of r^m terms by repeated application of the distributive laws that the r^m terms can be grouped in subsets

$$\{x_{\alpha(1)}x_{\alpha(2)}\cdots x_{\alpha(m)},\, x_{\alpha(2)}\cdots x_{\alpha(m)}x_{\alpha(1)},\,\cdots,\, x_{\alpha(\pi)}\cdots x_{\alpha(m)}x_{\alpha(1)}\cdots x_{\alpha(\pi-1)}\}$$
$$(\alpha:\ \{1,2,\cdots,m\}\to\{1,2,\cdots,r\})$$

where each later term is obtained by cyclic permutation of the factors of its predecessor and where the «period» π of the subset is the smallest natural number for which

$$\alpha(i)=\alpha(i+\pi),\ (1\leqq i\leqq m-\pi);\ \alpha(i)=\alpha(i+\pi-m),\ (m-\pi<i\leqq m).$$

He also observed that π divides m and that $\pi=1$ occurs precisely r times viz. for the subsets $\{x_1^m\},\cdots,\{x_r^m\}$.

We apply E. Landau's astute observation to the sum (3) of $|G|$ summands $\lambda(g)g$ focussing on those product terms

$$\lambda(\alpha(1))\cdots\lambda(\alpha(m))\alpha(1)\alpha(2)\cdots\alpha(m)\quad(\alpha:\ \{1,2,\cdots,m\}\to G)$$

for which the group element $\alpha(1)\alpha(2)\cdots\alpha(m)$ is equal to 1_G. Since the basic cyclic permutation of a product of group elements is achieved by application of the inner transformation by the first factor it follows that the Landau subsets either consist entirely of terms contained in $I1_G$ or they contain no non zero term of $I1_G$ at all. In the first event the sum of the corresponding terms is divisible by the period π in I.

Supposing m is a power of the prime number p occurring in Lemma 3 then the coeficient of 1_G in the presentation of a^m as a linear combination of G over I, say, the element λ_m of I, is congruent to $\Sigma_{g\in G\,\&\,g^m=1_G}\lambda(g)^m$ (modulo pI).

There holds the well known congruence

$$\alpha^m+\beta^m\equiv(\alpha+\beta)^m\pmod{pI}$$

for any two elements α, β of I as a consequence of the binomial theorem, hence

$$\lambda_m\equiv\Sigma_{\substack{g\in G\\ g^m=1_G}}\lambda(g))^m\pmod{pI}. \tag{8}$$

On the other hand it follows from Lemma 1 and from the assumptions made about a that $\lambda_m=0$ if $m\,|\,p^{n-1}$ and that $\lambda_{p^v}=1$.

Hence

$$(\Sigma_{\substack{g\in G\\|g|=p^v}}\lambda(g))^{p^v}\equiv\Sigma_{\substack{g\in G\\|g|=p^v}}\lambda(g)^{p^v}$$
$$\equiv\lambda_{p^{v-1}}^p+\Sigma_{\substack{g\in G\\|g|=p^v}}\lambda(g)^{p^v}$$
$$\equiv\Sigma_{\substack{g\in G\\g^{p^{v-1}}=1_G}}\lambda(g)^{p^v}+\Sigma_{\substack{g\in G\\|g|=p^v}}\lambda(g)^{p^v}$$
$$\equiv\Sigma_{\substack{g\in G\\g^{p^v}=1_G}}\lambda(g)^{p^v}$$
$$\equiv\lambda_{p^v}\equiv1\pmod{pI}\qquad\text{q. e. d.}$$

Corollary 1. *The torsion center of* $U(I[G])/U(I)1_G$ *is represented modulo* $U(I)/1_G$ *by the center elements of* G. *If an element* (3) *of* $I[G]$ *represents modulo* $U(I)1_G$ *a non central torsion element of* $U(I[G])/U(I)1_G$ *then its regular trace is zero.*

Proof. Apply the theorem to ag^{-1} for g of G subject to the non equation $\lambda(g)\neq0$.

Corollary 2. *Under the same assumptions the group ring of* G *over* I *contains no idempotent other than* 1_G.

Proof. Otherwise one would be able to replace I by its integral closure in $F(\zeta)$ where ζ is a primitive m-th root of unity with m not dividing the order of G such that one would have an idempotent e of $I[G]$ distinct from 1_G and that $\zeta e+1-e$ would be of order m modulo $U(I)1_G$ contrary to the theorem.

REFERENCES

[1] BERMAN, S. D., *On certain properties of integral group rings.* Dokl. Akad. Nauk. SSSR. **91**, 7-9 (1953).
[2] E. LANDAU, *Über die Irreduzibilität der Kreisteilungsgleichung.* Mathematische Zeitschrift **29**, 462-465.
[3] SAKSONOV, *Group rings of finite groups, I.* Publ. Math. Debrecen **18** (1971) 187-209.

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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BEMERKUNGEN ZU MEINER ARBEIT: «AN ALMOST SUBIDEMPOTENT RADICAL PROPERTY OF RINGS» (*)

VON

FERENC ANDOR SZÁSZ
Budapest

Dem Herrn Prof. Dr. ALMEIDA COSTA zu seinem 70-sten Geburtstag gewidmet

1. Für die nötigen Grundbegriffe verweisen wir auf A. ALMEIDA COSTA [5], N. DIVINSKY [6], N. JACOBSON [7], A. KERTÉSZ [8] und L. RÉDEI [12], weiterhin Verfasser [14].

2. Jeder hier betrachtete Ring wird assoziativ (i. a. nichtkommutativ und ohne Einselement) angenommen. Für beliebige additive Untergruppen B und C eines Ringes A, sei das Produkt, wie üblich ist, die durch sämtliche Elemente $b \cdot c$ mit $b \in B$ und $c \in C$ erzeugte Untergruppe von A^+. Weiterhin sei $R(a) = (a)_l \cdot A = aA + AaA$, dass ein zweiseitiges Ideal von dem Ring A ist, und hier $(a)_l$ bezeichnet das durch a in A erzeugte Hauptlinksideal. Das links-rechts Dual von $R(a)$ sei $R'(a) = Aa + AaA$. Entsprechend der Arbeit [15] des Verfassers, die solche Ringe, deren alle homomorphe Bilder keinen von Null verschiedenen Linksannihilator haben, wurden von mir als sogenannte E_5-Ringe bezeichnet, und diese Ringe sind nützbar Kriteria anzugeben, damit der Ring A ein zweiseitiges Einselement habe. (Sehe Klammer in Teil 6). (Verfasser hat in [15] auch die E_5-Ringe für $i = 0, 1, 2, 3, 4, 5$ und in [20] auch für $i = 6$ untersucht). Ein Ring A ist dann und nur dann ein Ω-Ring, wenn $a \in R(a)$ für jedes $a \in A$ und für das obige Ideal $R(a)$ gilt.

(*) Recebido em 2 de Maio de 1974.

3. Die E_5-Ringe habe ich mit 7 äquivalenten Bedingungen charakterisiert:

(i) A ist ein E_5-Ring;

(ii) $L = LA$ gilt für jedes Linksideal L von A;

(iii) $(a)_l = (a)_l A$ für jedes Hauptlinksideal $(a)_l$ von A;

(iv) $a \in R(a)$ für jedes $a \in A$;

(v) $(a) = R(a)$ für jedes zweiseitige Hauptideal (a) von A;

(vi) $I = IA$ für jedes Ideal I von A;

(vii) $(a) = (a)A$ gilt für jedes Hauptideal (a) von A.

4. Wir bemerken, dass Professor Gábor Szász [21] hat früher die Halbgruppen S mit $I = I \cdot S$, für jedes Ideal I von S untersucht.

5. Die Relation $a \in R(a)$ definiert eine F-Regularität im Sinne von B. Brown-N. H. McCoy [4]. Hierbei ist F eine Abbildung des Ringes A in die Menge aller Ideale von A, derart, dass bei beliebigem Ringhomomorphismus φ die Gleichung $(F(a))\varphi = F(a\varphi)$ gilt, und im Falle $a \in F(a)$ heisst a ein F-reguläres Element. Ein Ideal I ist F-regulär, wenn jedes sein Element in A (aber i. a. nicht in sich I selbst) F-regulär ist. Ist aber die Klasse der Ringe A mit $a \in F(a)$ für jedes $a \in A$ *erblich* bezüglich der Ideale, so ist das in A definierte F-Regularität mit der im Ideal I genommenen F-Regularität äquivalent. Das maximale F-reguläre Ideal $\mathrm{rad}_F A$ von A ist ein Radikal im Sinne von J. M. Maranda [10] und G. Michler [11]. Für eine beliebige homomorph abgeschlossene Klasse K von Ringen, bezeichne $\overline{K}$ das Amitsur-Kuroschsche untere allgemeine Radikalklasse (siehe N. Divinsky [6]). Ist R die Klasse aller E_5-Ringe, so erhält man $\overline{R}(A) \subseteq \mathrm{rad}_R A$ für jeden Ring A.

6. Mein lieber Kollega, Cornelis Ross (Holland), hat liebenswürdig bemerkt, dass im allgemeinen $R(A) \neq \mathrm{rad}_R A$ gilt, und er hat dafür auch explizites Beispiel aus einer Ringkataloge gefunden. (Ubrigens geben meine Sätze 3.1.1, 3.1.2, 3.1.3, 3.1.4, 3.2.1, 3.2.2, 3.3.1, 3.3.2, 3.4.1 and 3.4.2 von [15] auch jetzt richtige Kriteria für die Existenz des Einselementes in einem beliebigen Ring an).

7. Das Hauptziel dieser Note ist «Theorem 2» aus meiner Arbeit [19] zu korrigieren. Ich habe nämlich ausgesprochen, und dafür gab ich auch einen elementenfreien, nur mit Idealen arbeitenden Beweis in meiner Arbeit [20], dass die Klasse der E_5-Ringe bzw. der E_6-Ringe eine Amitsur-Kuroschsche Radikalklasse bildet. Es gilt also $\overline{R} = R$. Aber der zweite Teil des «Theorems 2» und der Beweis des ersten Teiles des «Teorems 2» von [19] sieht nicht absolut korrekt zu sein, denn R ist bezüglich der Ideale nicht erblich. So z. B. der Ring Z der ganzen rationalen Zahlen gehört der Klasse R, aber sein Ideal $2Z$ schon nicht.

Auch verschiedene «Propositions» bleiben nach dem «Theorem 2» i. a. nur dann gültig, wenn mann immer $\mathrm{rad}_R A$ statt $R(A)$ nimmt. Proposition 7 gilt aber, und zwar auch in einer schärferen Gestalt, nämlich mit «lokal nilpotent» statt «nilpotent» (sein Beweis siehe in meinen Arbeiten [18] und [19]).

LITERATURVERZEICHNIS

[1] S. A. Amitsur, *A general theory of radicals, II,* Amer. Jour. Math. **76** (1954) 100-125.

[2] W. A. Andrunakiewitsch, *Biregular rings,* Mat. Sbornlk N. S. **39** (81) (1956) 447-464 (Russisch).

[3] W. A. Andrunakiewitsch, *Radikale von assoziativen Ringen,* Mat. Sbornik N. S. **44** (86) (1958) 179-212.

[4] B. Brown-N. H. McCoy, *Radicals and subdirect sums,* Amer. Math. Jour. **69** (1947) 46-58.

[5] A. A. Costa, *Cours d'Algèbre Générale,* (Lisboa) 1969.

[6] N. Divinsky, *Rings and Radicals,* London-Ontoario (1965).

[7] N. Jacobson, *Structure of Rings,* Providence (USA, 1956, 1964).

[8] A. Kertész, *Vorlesungen über Artinsche Ringe,* Budapest (1968).

[9] A. G. Kurosch, *Radikale von Ringen und von Algebren,* Mat. Sbornik **33** (1952) 13-26.

[10] J. M. Maranda, *Injective structures,* Trans. Amer. Math. Soc. **110** (1964) 98-135.

[11] G. Michler, *Radikale und Sockel,* Math. Annalen **167** (1966) 1-48.

[12] L. Rédei, *Algebra, I.,* Leipzig (1959).

[13] C. Roos, *The radical property of nonassociative rings such that every homomorphic image has no nonzero left annihilating ideals,* Math. Nachrichten (im Erscheinen).

[14] F. Szász, *Radikale der Ringe,* Budapest (im Frscheinen, cca 600 Seiten).

[15] F. Szász, *Einige Kriterien für die Existenz des Einselementes in einem Ring,* Acta Sci. Math. Szeged **28** (1967) 31-37.

[16] F. Szász, *Reduktion eines Problems bezüglich der* Brown-McCoy*schen Radikalringe,* Acta Sci. Math. Szeged **31** (1970) 167-172.

[17] F. SZÁSZ, *The radical property of rings such that every homomorphic image has no nonzero left annihilators*, Math. Nachrichten **48** (1971) 371-375.
[18] F. SZÁSZ, Remarks on my paper *The radical property of rings such that every homomorphic image has no nonzero left annihilators*, Math. Nachrichten (im Erscheinen).
[19] F. SZÁSZ, *An almost subidempotent radical property of rings*, Colloquia Mathematica Societatis János Bolyai 6, Rings, Modules, Radicals, Keszthely (Hungary, Balaton Lake, 1971) 483-505.
[20] F. SZÁSZ, *A second almost subidempotent radical for rings*, Math. Nachrichten (im Erscheinen).
[21] G. SZÁSZ, *Uber Halbgruppen, die ihre Ideale reproduzieren*, Acta Sci. Math Szeged **27** (1966) 141-145.

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
Lisboa, 1974

SUR LA MESURE DE CARATHÉODORY(*)

PAR

J. VICENTE GONÇALVES

Au Prof. ALMEIDA COSTA

Le concept de *partition métrique* de l'espace, dont il est question ici, dissimule autant que possible la raideur de la définition de Carathéodory (1) et dans bien des cas nous épargne la peine d'avoir à remonter jusqu'à cette définition pour établir la mesurabilité d'un ensemble donné.

Toute partition de l'espace X,

1) $$X = \Sigma X_i = X_1 + X_2 + \cdots \text{ (2)},$$

détermine dans chaque ensemble $A \subset X$ une partition parallèle

2) $$A = A X = \Sigma A X_i ;$$

et, une mesure extérieure μ^* étant définie dans X, il se peut que cette partition 2) entraîne à son tour

3) $$\mu^* A = \Sigma \mu^* A X_i ,$$

quel que soit $A \subset X$. S'il en est bien ainsi, on dit alors que 1) est une *partition métrique* de l'espace par rapport à la mésure extérieure μ^*.

(*) Recebido em 7 de Novembro de 1974.

(1) «It is rather difficult to get an intuitive understanding of the meaning of μ^*-mesurability...», Halmos, *Mesure Theory* (1950), p. 44.

(2) L'emploi de $+$ et Σ comme signes d'union de certains ensembles signifie que ceux-ci sont supposés disjoints.

Pour deux composantes seulement, E et E', les relations précédentes se reduisent à

1_0) $X = E + E'$,

2_0) $A = AE + AE'$,

3_0) $\mu^* A = \mu^* AE + \mu^* AE'$.

Ainsi qu'il est bien connu, c'est la vérification ou non vérification de 3_0) pour tout $A \subset X$ qui caractérise E comme ensemble mesurable ou non mesurable par rapport à la mesure extérieure μ^*. C'est la définition de Carathéodory, la seule dont'il sera question ici.

Lorsque 3_0) se trouve vérifiée pour tout $A \subset X$, en y remplaçant A par $A(E+U)$ (donc $U \subset E'$), on a

4) $$\mu^* A(E+U) = \mu^* AE + \mu^* AU,$$

quels que soient $A \subset X$ et $U \subset E'$.

Réciproquement, 4) étant vérifiée dans ces conditions, en y faisant $U = E'$ on retrouve 3_0) pour tout $A \subset X$.

On peut donc prendre indifféremment 3_0) ou 4) pour caractériser E comme ensemble mesurable ou non mesurable par rapport à μ^* (1).

Mais 4), par sa souplesse et haute teneur en nécessité, que lui prêtent les ensembles A et U, se montre souvent plus utile que l'équation originale elle-même.

Dans ce qui va suivre les mots «métrique» et «mesurable» si rapportent toujours à la mésure extérieure μ^*.

Théorème I. *Chaque ensemble mesurable peut prendre place dans une partition métrique.*

Parce que 3_0) fait de 1_0) une telle partition.

Théorème II. *Chaque ensemble appartenant à une partition métrique est mesurable.*

(1) L'idée de remplacer A par $A(E \cup F)$ dans 3_0) se trouve dans Halmos, *Mesure Theory*, p. 45, (e). Carathéodory s'est restreint à

$$\mu^*(E+U) = \mu^* E + \mu^* U$$

)*Reelle Funktionnen*, p. 253).

Pour plus de simplicité, envisageons X_1 dans 1), la condition 3) de métricité étant supposée satisfaite.

De par la sous-additivité de μ^*, on a

$$\mu^* A = \mu^* A(X_1 + X'_1) \leqq \mu^* AX_1 + \mu^* AX'_1,$$

$$\mu^* AX'_1 = \mu^* A(X_2 + \cdots) \leqq \mu^* AX_2 + \cdots$$

et ceci n'entraîne 3) que si $\leqq$ signifie partout $=$. La métricité de 1) assure donc la mesurabilité de chaque composante $X_1, X_2, \cdots$.

D'après I et II, le concept d'ensemble mesurable et celui de composante d'une partition métrique sont tout à fait équivalents.

Le recours à 4) et l'emploi de partitions métriques permettent d'abréger le traitement de diverses questions de mesure.

Suivent quelques exemples.

1. MESURE EXTÉRIEURE DE $A(E \cup F)$, E ÉTANT SUPPOSÉ MESURABLE.

D'après 4),

$$\mu^* A(E \cup F) = \mu^* A(E + E'F) = \mu^* AE + \mu^* AE'F;$$

de même,

$$\mu^* AF = \mu^* AF(E+E') = \mu^* AEF + \mu^* AE'F.$$

À moins qu'on ait $\mu^* AEF = +\infty$, ce sera donc

5) $$\mu^* A(E \cup F) = \mu^* AE + \mu^* AF - \mu^* AEF.$$

2. MESURABILITÉ DE $E'F$, $E'F'$ ET $E \cup F$, EN SUPPOSANT E ET F MESURABLES.

Puisque

6) $$X = E + E'(F+F') = E + E'F + E'F',$$

on a par 4)

$$\mu^* A = \mu^* A E + \mu^* A E' (F + F') =$$
$$= \mu^* A E + \mu^* A E' F + \mu^* A E' F',$$

ce qui établit la métricité de 6) (II). $E' F = F - E$ et $E' F' = F' - E$ sont donc mesurables et il en est de même de $E \cup F = (E' F')'$.

3. ADDITIVITÉ POUR $Y = \Sigma Y_i$, LES Y_i (EN NOMBRE FINI OU INFINI) ÉTANT MESURABLES.

Posons

$$Y^i = Y_i + Y_{i+1} + \cdots, \quad Z = X - Y.$$

Nous aurons

$$X = Y + Z = Y_1 + (Z + Y^2),$$
$$\cdots\cdots\cdots\cdots$$
$$Y^i + Z = Y_i + (Z + Y^{i+1}),$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$

Les Y_i étant mesurables, il en résulte par 4)

$$\mu^* A = \sum_1^n \mu^* A Y_i + \mu^* A (Z + Y^{n+1})$$
$$\geqq \mu^* A Z + \sum_1^n \mu^* A Y_i,$$

donc (μ^* étant sous-additive)

$$\mu^* A = \mu^* A Z + \Sigma \mu^* A Y_i,$$

ce qui assure la métricité de $X = Z + \Sigma Y_i$ et par là la mesurabilité de Z et $Y = Z'$. Et de $X = Z + Y$ on tire enfin

$$\mu^* A Y = \Sigma \mu^* A Y_i.$$

En mettant Y à la place de E dans 4), on trouve

$$\mu^* A (U + \Sigma Y_i) = \mu^* A U + \Sigma \mu^* A Y_i,$$

résultat dû à J. J. Dionísio, qui s'en est servi pour établir la σ-additivité.

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
Lisboa, 1974

SOBRE OS SEMI-ANÉIS-μ_d (*)

POR

A. J. ANTUNES MONTEIRO (**)

Ao Senhor Professor ALMEIDA COSTA

1. INTRODUÇÃO

1.1. Definição: Seja $\mathfrak{S}$ um semi-anel. Uma família $\mathfrak{M}^d = \{\mathfrak{r}_\alpha\}_{\alpha \in A}$, de ideais direitos de $\mathfrak{S}$, diz-se um sistema-μ_d quando, tomados arbitrariamente $\alpha, \beta \in A$, existe $\mathfrak{h}$, ideal direito de $\mathfrak{S}$, tal que $\mathfrak{r}_\alpha \mathfrak{h} \mathfrak{r}_\beta \in \mathfrak{M}^d$. Uma família $\mathfrak{P}^d = \{\mathfrak{r}_\alpha\}_{\alpha \in A}$, de ideais direitos de $\mathfrak{S}$, diz-se um sistema-π_d quando, para cada $\alpha \in A$, existe um ideal direito $\mathfrak{h}$, de $\mathfrak{S}$, tal que $\mathfrak{r}_\alpha \mathfrak{h} \mathfrak{r}_\alpha \in \mathfrak{P}^d$.

Substituindo, nestas definições, «ideal direito» por «ideal esquerdo», falaríamos, respectivamente, em sistemas-μ_e e -π_e.

1.1. a Observações: 1.ª) Poder-se-ia, naturalmente, falar em sistemas de ideais direitos que fossem sistemas-μ_e, no seguinte sentido: dados dois ideais direitos, digamos $\mathfrak{r}$ e $\mathfrak{r}'$, do sistema em causa, existiria um ideal esquerdo $\mathfrak{e}$ tal que $\mathfrak{r}\mathfrak{e}\mathfrak{r}'$ estivesse ainda entre os ideais direitos considerados. De forma análoga, poder-se-iam fazer diversas combinações de «direito» e «esquerdo».

2.ª) Todo o sistema-μ_d (resp.: -μ_e) é um sistema-π_d (resp.: -π_e). O conjunto de todos os ideais direitos (resp.: esquerdos) de $\mathfrak{S}$ é um sistema-μ_d (resp.: -μ_e).

1.2 Teorema: *Seja* $\Omega = \{\mathfrak{a}_\alpha\}_{\alpha \in A}$ *uma família de ideais (bilaterais) de* $\mathfrak{S}$, *de tal modo que* Ω *seja simultaneamente um sistema-*μ_d *(quando considerado como família de ideais direitos) e um sistema-*μ_e

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(quando considerado como família de ideais esquerdos). Então, Ω é um sistema-μ. Reciprocamente, todo o sistema-μ de ideais de $\mathfrak{S}$ é simultaneamente um sistema-μ_d e um sistema-μ_c.

Observação: Para a definição de «sistema-μ», ver [1].

Demonstração: Tomados $\alpha, \beta \in A$, seja $\mathfrak{r}$ um ideal direito de $\mathfrak{S}$ tal que $\mathfrak{a}_\alpha \mathfrak{r} \mathfrak{a}_\beta = \mathfrak{a}_\gamma \in \Omega$ e seja $\mathfrak{e}$ um ideal esquerdo de $\mathfrak{S}$ tal que $\mathfrak{a}_\alpha \mathfrak{e} \mathfrak{a}_\gamma \in \Omega$.

Nestas condições, vemos que: $\mathfrak{a}_\alpha \mathfrak{e} \mathfrak{a}_\alpha \mathfrak{r} \mathfrak{a}_\beta \in \Omega$.

O teorema está demonstrado, já que a parte recíproca é óbvia,

c. q. d.

1.3 Definição: Um sistema-μ_d, digamos $\mathfrak{M}_0^d$, diz-se particular quando $\mathfrak{M}_0^d \subseteq C^d(\mathfrak{U}_0^d)$, em que $\mathfrak{U}_0^d = \bigcup\limits_{\substack{\mathfrak{r} \notin \mathfrak{M}_0^d \\ \mathfrak{r} \text{ é ideal direito de } \mathfrak{S}}} \mathfrak{r}$ e em que, para cada $X \subseteq \mathfrak{S}$, $C^d(X)$ designa o conjunto dos ideais direitos de $\mathfrak{S}$ que não estão contidos em X. Do mesmo modo se define «sistema-π_d particular».

1.3.a Observações: 1.ª) As definições aqui apresentadas e os diversos teoremas que delas resultam, podem, naturalmente, converter-se noutras e noutros, em que figurem os conceitos de «-μ_e» e «-π_e», em vez de «-μ_d» e «-π_d».

2.ª) Dada a analogia entre os raciocínios aqui desenvolvidos e os apresentados em [3], omitiremos as demonstrações que das correspondentes demonstrações que figuram em [3] se possam deduzir.

De acordo com 1.3.a, enunciaremos, sem as provarmos, as seguintes proposições:

1.4 Teorema: *Todo o sistema-π_d é união de sistemas-μ_d numeráveis. De forma recíproca, qualquer união de sistemas-μ_d é um sistema-π_d.*

1.5 Teorema: *Se $\mathfrak{M}_0^d$ é um sistema-μ_d particular, então $\mathfrak{M}_0^d = C^d(\mathfrak{U}_0^d)$. Se $\mathfrak{P}_0^d$ é um sistema-π_d particular, então $\mathfrak{P}_0^d = C^d(\mathfrak{U}_0^d)$. O conjunto vazio e o conjunto de todos os ideais direitos não vazios de $\mathfrak{S}$ são sistemas-μ_d particulares.*

1.6 Teorema: *Uma união de sistemas-π_d particulares é um sistema-π_d particular.*

Apresentaremos agora a seguinte definição:

1.7 Definição: Seja $\mathfrak{S}$ um semi-anel e $X \subseteq \mathfrak{S}$. Diz-se que X é primo-*d* quando, sendo $\mathfrak{r}_1$ e $\mathfrak{r}_2$ ideais direitos de $\mathfrak{S}$, de $\mathfrak{r}_1 \mathfrak{r}_2 \subseteq X$ se conclua $\mathfrak{r}_1 \subseteq X$ ou $\mathfrak{r}_2 \subseteq X$. Diz-se que X é semi-primo-*d* quando, sendo $\mathfrak{r}$ um ideal direito de $\mathfrak{S}$, de $\mathfrak{r}^2 \subseteq X$ se conclua $\mathfrak{r} \subseteq X$. Analogamente se definem conjuntos «primos-*e*» e «semi-primos-*e*».

De novo, segundo o critério anunciado em 1.3.a, enunciaremos as seguintes proposições:

1.8 Teorema: *Se $\mathfrak{p}$ é um ideal direito primo-d, $C^d(\mathfrak{p})$ é um sistema-μ_d particular. Se $\mathfrak{p}$ é semi-primo-d, $C^d(\mathfrak{p})$ é um sistema-π_d particular.*

1.9 Teorema: *Seja $\mathfrak{P}_0^d = C^d(\mathfrak{U}_0^d)$ um sistema-π_d particular e $\mathfrak{r}$ um ideal direito de $\mathfrak{S}$, tal que $\mathfrak{P}_0^d \subseteq C^d(\mathfrak{r})$. Então, existem ideais direitos $\mathfrak{g}$ maximais tais que $\mathfrak{r} \subseteq \mathfrak{g}$ e $\mathfrak{P}^d \subseteq C^d(\mathfrak{g})$. Estes ideais direitos maximais são semi-primos-d.*

1.10 Teorema: *Seja $\mathfrak{y} = \bigcup\limits_{j \in J} \mathfrak{y}_j$ uma união de ideais direitos. São equivalentes as afirmações seguintes:*

(i) *$\mathfrak{y}$ é um ideal semi-primo-d minimal contendo o ideal direito $\mathfrak{r}$.*

(ii) *$C^d(\mathfrak{y})$ é um sistema-π_d maximal entre os sistemas-π_d particulares contidos em $C^d(\mathfrak{r})$.*

Introduzimos agora o conceito de «semi-anel-μ_d»:

1.11 Definição: O semi-anel $\mathfrak{S}$ diz-se *um semi-anel-μ_d* quando, dado um sistema-μ_d, $\mathfrak{M}^d$, e uma cadeia $\{\mathfrak{r}_\lambda\}_{\lambda \in L}$ de ideais direitos de $\mathfrak{S}$, a hipótese de ser $\mathfrak{M}^d \subseteq C^d(\mathfrak{r}_\lambda)$, para cada $\lambda \in L$, arrasta que que se tenha $\mathfrak{M}^d \subseteq C^d\left(\bigcup\limits_{\lambda \in L} \mathfrak{r}_\lambda\right)$. Analogamente se definem «semi-anel-μ_e», «-π_d» e «-π_e».

1.11 a Observações: Todo o semi-anel que verifique a condição de cadeia ascendente para os seus ideais direitos é um semi-anel-μ_d, o mesmo acontecendo a todo o semi-anel-π_d. Todo o semi-anel-μ_d ou -μ_e (resp.: -π_d ou -π_e) é um semi-anel-μ (resp.: -π).

1.12 Teorema: *Todo o semi-anel-μ_d é um semi-anel-π_d. Este resultado deve comparar-se com uma das observações contidas em* 1.1. a. (Nota: Ver também [2]).

1.13 Teorema: *Seja $\{\mathfrak{X}_\sigma\}_{\sigma \in S}$ uma cadeia de ideais direitos semi-primos*-d *de um semi-anel $\mathfrak{S}$. Ponhamos $X = \{\mathfrak{r} \mid \mathfrak{r}$ é ideal direito de $\mathfrak{S}$ e existe $\sigma \in S$ tal que $\mathfrak{r} \subseteq \mathfrak{X}_\sigma\}$. Então:*

(i) *Se $\mathfrak{r}, \mathfrak{r}' \in X$, também $(\mathfrak{r}, \mathfrak{r}') \in X$.*

(ii) *Se $\mathfrak{r} \in X$ e $\mathfrak{r}'$ é um ideal direito de $\mathfrak{S}$ tal que $\mathfrak{r}' \subseteq \mathfrak{r}$, também $\mathfrak{r}' \in X$.*

(iii) *O conjunto dos ideais direitos de $\mathfrak{S}$, não vazios e não pertencentes a X, é um sistema-π_d.*

1.14 Teorema: *Seja $\mathfrak{S}$ um semi-anel-μ_d e $G = \{\mathfrak{y}_\alpha\}_{\alpha \in A}$ uma família não vazia de ideais direitos que satisfaça as propriedades* (i), (ii) *e* (iii) *de* 1.13. *Então, $\mathfrak{y} = \bigcup_{\alpha \in A} \mathfrak{y}_\alpha \in G$ e $C^d(\mathfrak{y}) = \mathfrak{P}_0^d$ é um sistema-π_d particular, que, por* (iii), *coincide com o complementar de G no conjunto dos ideais direitos não vazios de $\mathfrak{S}$.*

1.15 Teorema: *Seja $\mathfrak{S}$ um semi-anel-μ_d e $\{\mathfrak{X}_\sigma\}_{\sigma \in S}$ uma cadeia de ideais direitos semi-primos*-d. *Nestas condições:*

(i) *Existe $\tau \in S$ tal que* $\mathfrak{X}_\tau = \bigcup_{\sigma \in S} \mathfrak{X}_\sigma$

(ii) $C^d\left(\bigcup_\sigma \mathfrak{X}_\sigma\right) = \bigcap_\sigma C^d(\mathfrak{X}_\sigma)$.

1.15. a Corolário: *Um semi-anel-μ_d verifica a condição de cadeia ascendente para os seus ideais semi-primos*-d.

1.16 Teorema: *Seja $\mathfrak{r}$ um ideal direito do semi-anel-μ_d $\mathfrak{S}$ e $\mathfrak{P}^d$ um sistema-π_d maximal contido em $C^d(\mathfrak{r})$. Então, $\mathfrak{P}^d$ é um sistema-π_d particular.*

2. TEOREMA FUNDAMENTAL

Teorema: *O semi-anel $\mathfrak{S}$ é um semi-anel-μ_d se e só se verifica as duas seguintes condições:*

(i) *Se $\mathfrak{r}$ é um ideal direito e $\mathfrak{P}^d$ um sistema-π_d maximal contido em $C^d(\mathfrak{r})$, então $\mathfrak{P}^d$ é um sistema-π_d particular.*

(ii) *$\mathfrak{S}$ verifica a condição de cadeia ascendente para os seus ideais semi-primos*-d.

A «necessidade» destas condições está expressa em 1.15. a *e* 1.16. *Quanto à demonstração da respectiva «suficiência», omiti-la-emos, de acordo com* 1.3. a.

3. ALGUMAS OBSERVAÇÕES SOBRE ANEIS-μ

Deve observar-se que tudo quanto se fez acima se aplica ao caso particular em que $\mathfrak{S}$ é um anel. Por outro lado, os raciocínios que apresentaremos a seguir podem desenvolver-se do mesmo modo num contexto de anéis-μ_d ou -μ_e. Por comodidade, porém, apresentá-los-emos referidos a anéis-μ.

3.1 Teorema: *Seja $\mathfrak{H}$ um ideal de um anel $\mathfrak{A}$ e $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{H}$ o epimorfismo canónico. Seja $\mathfrak{M}$ um sistema-μ em $\mathfrak{A}$ e ponhamos $\mathfrak{M}' = \{\pi(\mathfrak{a}) \mid \mathfrak{a} \in \mathfrak{M}\}$. Então, $\mathfrak{M}'$ é um sistema-μ em $\mathfrak{A}/\mathfrak{H}$.*

Demonstração: Tomando ideais $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$ e $\mathfrak{y}$ tal que $\mathfrak{a}\mathfrak{y}\mathfrak{b} \in \mathfrak{M}$, vemos que $\pi(\mathfrak{a})\pi(\mathfrak{y})\pi(\mathfrak{b}) = \pi(\mathfrak{a}\mathfrak{y}\mathfrak{b}) \in \mathfrak{M}'$.

c. q. d.

3.2 Teorema: *Seja $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{H}$ como em 3.1 e suponhamos que $\mathfrak{H}^2 = \mathfrak{H}$. Seja $\mathfrak{M}'$ um sistema-μ em $\mathfrak{A}/\mathfrak{H}$ e ponhamos*

$$\mathfrak{M} = \{\pi^{-1}(\mathfrak{a}') \mid \mathfrak{a}' \in \mathfrak{M}'\}.$$

Então, $\mathfrak{M}$ é um sistema-μ em $\mathfrak{A}$.

Demonstração: Tomemos, em $\mathfrak{A}/\mathfrak{H}$, ideais $\mathfrak{a}', \mathfrak{b}' \in \mathfrak{M}'$ e $\mathfrak{y}'$ tal que $\mathfrak{a}'\mathfrak{y}'\mathfrak{b}' \in \mathfrak{M}'$. Ponhamos $\mathfrak{a} = \pi^{-1}(\mathfrak{a}')$, $\mathfrak{b} = \pi^{-1}(\mathfrak{b}')$, $\mathfrak{y} = \pi^{-1}(\mathfrak{y}')$.
O facto de ser $\mathfrak{H} = \mathfrak{H}^2 = \mathfrak{H}^3 \subseteq \mathfrak{a}\mathfrak{H}\mathfrak{b}$ mostra que $\pi^{-1}(\mathfrak{a}'\mathfrak{y}'\mathfrak{b}') = \mathfrak{a}\mathfrak{y}\mathfrak{b}$, o que demonstra a tese.

c. q. d.

3.2. a Corolário: *Num epimorfismo de anéis, de núcleo idempotente, os sistemas-μ correspondem-se, segundo os processos indicados em* 3.1 *e* 3.2.

3.3 Teorema: *Com a notação de* 3.2, *a condição* $\mathfrak{H}^2 = \mathfrak{H}$ *é necessária para que a imagem inversa (de acordo com* 3.2) *de um sistema-μ seja um sistema-μ.*

Demonstração: Em $\mathfrak{A}/\mathfrak{H}$, $\mathfrak{M} = \{\{\mathfrak{H}\}\}$ é um sistema-μ. Neste caso, $\mathfrak{M} = \{\mathfrak{H}\}$. Por hipótese, $\mathfrak{M}$ é um sistema-μ, pelo que existe em $\mathfrak{A}$ um ideal $\mathfrak{h}$ tal que $\mathfrak{H}\mathfrak{h}\mathfrak{H} = \mathfrak{H}$.

Assim: $\mathfrak{H} = \mathfrak{H}\mathfrak{h}\mathfrak{H} \subseteq \mathfrak{H}\mathfrak{H} = \mathfrak{H}^2$, pelo que $\mathfrak{H}^2 = \mathfrak{H}$.

c. q. d.

3.3. a Corolário: *Num epimorfismo de anéis, os sistemas-μ correspondem-se, segundo os processos indicados em* 3.1 *e* 3.2, *se e só se o núcleo é um ideal idempotente.*

3.4 Teorema: *Num epimorfismo de núcleo idempotente, a imagem de um anel-μ é ainda um anel-μ.*

Demonstração: Seja $\mathfrak{A}$ um anel-μ, $\mathfrak{H}$ um ideal idempotente de $\mathfrak{A}$ e $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{H}$ o epimorfismo canónico.

Seja $\{\mathfrak{a}'_\lambda\}_{\lambda \in L}$ uma cadeia de ideais em $\mathfrak{A}/\mathfrak{H}$ ($\{\pi^{-1}(\mathfrak{a}'_\lambda)\}_{\lambda \in L}$ será uma cadeia de ideais em $\mathfrak{A}$) e $\mathfrak{M}'$ um sistema-μ em $\mathfrak{A}/\mathfrak{H}$. Por 3.3. a, $\mathfrak{M} = \{\pi^{-1}(\mathfrak{a}') \mid \mathfrak{a}' \in \mathfrak{M}'\}$ é um sistema-μ em $\mathfrak{A}$.

Suponhamos que, para cada $\lambda \in L$, $\mathfrak{M}' \subseteq C(\mathfrak{a}'_\lambda)$. Então, para cada $\lambda \in L$, $\mathfrak{M} \subseteq C(\pi^{-1}(\mathfrak{a}'_\lambda))$.

Para vermos que $\mathfrak{M}' \subseteq C\left(\bigcup_\lambda \mathfrak{a}'_\lambda\right)$, suponhamos que tal não acontece e seja $\mathfrak{b}' \in \mathfrak{M}'$, $\mathfrak{b}' \subseteq \bigcup_\lambda \mathfrak{a}'_\lambda$.

Então, $\pi^{-1}(\mathfrak{b}') \subseteq \pi^{-1}\left(\bigcup_\lambda \mathfrak{a}'_\lambda\right) = \bigcup_\lambda \pi^{-1}(\mathfrak{a}'_\lambda)$ e $\pi^{-1}(\mathfrak{b}') \in \mathfrak{M}$, o que é absurdo, pois, por $\mathfrak{A}$ ser um anel-ν, o que se viu acima arrasta que $\mathfrak{M} \subseteq C\left(\bigcup_\lambda \pi^{-1}(\mathfrak{a}'_\lambda)\right)$.

c. q. d.

3.4. a Observação: A afirmação recíproca de 3.4 não é verdadeira. Com efeito, seja $\mathfrak{A}$ um anel-μ tal que $\mathfrak{A}^2 \neq \mathfrak{A}$. Tem-se $\mathfrak{A}/\mathfrak{A} = (0)$. Há, pois, um epimorfismo $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}$ em que $\mathfrak{A}$ e $\mathfrak{A}/\mathfrak{A}$ são anéis-μ, mas em que o núcleo (que é $\mathfrak{A}$) não é idempotente.

BIBLIOGRAFIA

[1] Almeida Costa, A., *μ-systèmes et π-systèmes d'idéaux.* Rev. Fac. Ci. Lisboa, **7** (1959), 235-243.

[2] Noronha Galvão, M.ª L., *Sobre a teoria de* Noether-Krull *em semi-anéis.* Rev. Fac. Ci. Lisboa, **8** (1961), 175-256.

[3] Almeida Costa, A., *Les μ-demi-anneaux.* Séminaire Dubreil-Pisot, Journées d'Algèbre (1972), Paris.

ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
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ON PARACOMPACT DIFFERENTIABLE MANIFOLDS (*)

BY

J. J. DIONÍSIO (**)

Dedicated to Prof. A. ALMEIDA COSTA

ABSTRACT. In this paper we give some sufficient conditions for the paracompactness of a differentiable manifold.

Let us consider a C^∞ differentiable manifold M. We define a *partition of unity* on M to be a collection $\{\varphi_i\}$ of non-zero and non-negative differentiable functions $\varphi_i : M \to R$ on M such that: $C_i = \operatorname{supp} \varphi_i$ is compact and lies in some coordinate domain; the collection $\{C_i\}$ is locally finite; and $\sum_i \varphi_i = 1$.

We remember five equivalent conditions for the *paracompactness* of M:

(1) M admits a partition of unity.

(2) M is a Hausdorff manifold and any open covering of M admits a locally finite open refinement.

(3) M is a Hausdorff manifold with an atlas such that the domain of each chart intersects only a finite number of domains of other charts.

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(4) Each component of M (in the topology induced by the differentiable structure) is a HAUSDORFF subspace with countable basis for its topology.

(5) M admits a partition of unity subordinated to a given open covering (that is, the support of each function of the partition is contained in some open set of that covering).

The equivalence of these conditions is easily established by the chain of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. We also remember that M admits a finite partition of unity if and only if M is compact. All the partitions of unity are then finite.

The following well known result will be referred to as A: If M is a HAUSDORFF manifold and admits a positive-definite Riemannian structure, then M is paracompact, and conversely.

THEOREM 1. *Let* $\varphi : \mathrm{M} \to \mathrm{M}'$ *be differentiable, global and with injective differential* $\varphi_{*\mathrm{p}}$ *at each point* $\mathrm{p} \in \mathrm{M}$. *Let* M *be a* HAUSDORFF *and* M′ *a paracompact manifold. Then* M *is also paracompact.*

PROOF. M' admits a positive-definite Riemannian structure, by A. This metric induces on M, under φ, a positive-definite Riemannian structure. Then, by A, M is paracompact.

COROLLARY 1. *If* M *is submanifold of a paracompact manifold* M′, *then* M *is also paracompact.*

PROOF. In Theorem 1 take as φ the natural injection $j : M \to M'$. The manifold M' being paracompact, it is HAUSDORFF and so its submanifold M is also HAUSDORFF. Hence, by Theorem 1, M is is paracompact.

COROLLARY 2. *If* M *is a connected submanifold of a paracompact manifold* M′, *then* M *admits a countable basis.*

REMARK. Corollary 2 extends the following result in MATSUSHIMA [2], p. 99, Theorem 2: Let M' be a manifold with a countable basis, and let M be a connected submanifold of M'. Then M also has a countable basis.

PROOF OF COROLLARY 2. From Corollary 1 we infer that M is paracompact. But M is also connected. Hence, characterization (4) of paracompactness shows that M has a countable basis.

THEOREM 2. *Let* $\varphi : \mathrm{M}' \to \mathrm{M}$ *be differentiable, global, surjective and with surjective differential* $\varphi_{*\mathrm{p}}$ *at each point* $\mathrm{p} \in \mathrm{M}$. *If* M′ *is* HAUSDORFF *and admits a positive-definite Riemannian structure, projectable under* φ, *then* M *is paracompact.*

PROOF. The metric on M' induces through $\varphi : M' \to M$ a positive-definite Riemannian structure on M. (See BRICKELL-CLARK [1], p. 164, Proposition 9. 4. 3; the hypothesis $\dim M = \dim M'$ included in this Proposition is superfluous, according to a forthcoming paper of the author «Métricas riemannianas e submersões»). The topology of M is HAUSDORFF and $\varphi : M' \to M$ is an open mapp (as a submersion). The conditions of A are satisfied and hence M is paracompact.

THEOREM 3. *Let* G *be a transformation group acting freely and discontinuously in a* HAUSDORFF *manifold* M′. *Let* M′ *admit a positive-definite Riemannian structure which is invariant under the group* G. *Then, the structure of the quotient manifold* G|M′ *is paracompact.*

PROOF. The set quotient G/M' acquires, in the conditions above, a structure of quotient manifold M (with the same dimension as M'). The natural surjection $\varphi : M' \to M$ is differentiable and its differential φ_{*p} is surjective at each point $p \in M'$ (see BRICKELL-CLARK [1], p. 98, Proposition 6. 5. 1). The positive-definite Riemannian structure of M' is projectable under φ, because it is invariant under the group G (ibidem, p. 165, Proposition 9. 4. 4). Hence, Theorem 2 applies to show that $M = G/M'$ is paracompact.

COROLLARY. *Let* G *be a finite transformation group acting freely on a* HAUSDORFF *manifold* M′. *Let* M′ *admit a positive-definite Riemannian structure which is invariant under the group* G. *Then the structure of the quotient manifold* G/M′ *is paracompact.*

PROOF. The group G acts discontinuously, because M' is HAUSDORFF and G is finite (see BRICKELL-CLARK [1], p. 101, Problem **6**. 5. 4). Hence the conditions of Theorem 3 are satisfied and so the quotient manifold G/M' is paracompact.

REFERENCES

[1] F. BRICKELL and R. S. CLARK, *Differentiable Manifolds* (Van Nostrand, 1970).

[2] Y. MATSUSHIMA, *Differentiable Manifolds* (Marcell Dekker, 1972).

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ESTUDOS DE MATEMÁTICA em homenagem ao Prof. A. Almeida Costa
Lisboa, 1974

TRANSFORMAÇÕES ISOMÉTRICAS EM R^n SUA APLICAÇÃO À CINEMÁTICA DO SÓLIDO INVARIÁVEL (*)

POR

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Este trabalho representa um desenvolvimento dos assuntos que constituem habitualmente as minhas primeiras lições de Cinemática do sólido invariável. Compõe-se de duas partes: a primeira, que pertence ao domínio da Álgebra Linear, ocupa-se sobretudo do estudo das transformações isométricas em R^n, orientado no sentido de apresentar as conclusões de maior incidência na teoria dos deslocamentos dos sólidos. Na segunda, faz-se a aplicação dos resultados obtidos na primeira parte ao estudo destes deslocamentos.

Trata-se assim do que pode designar-se por Notas de Curso, nelas fazendo intervir um critério pessoal, quer na sua orientação, quer na sua elaboração. Por este facto, e ainda porque exemplificam, creio que bem, a forma como, no meu entender, se deve trabalhar em qualquer questão do domínio da Matemática Aplicada, julgo que possa ter interesse a sua publicação.

A. TRANSFORMAÇÕES ISOMÉTRICAS

1 — Aplicações lineares

Seja φ um homeomorfismo da classe C^2 de R^n sobre R^n. Seja $J(x)$ a matriz jacobiana de φ: $(J)_{ik} = \frac{\partial y_i}{\partial x_k}$, onde $y = \varphi(x)$. Para

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todo $h \in \{1, \cdots, n\}$ [1], ponha-se $J_h = \dfrac{\partial J}{\partial x_h}$. As matrizes J_h e J_h^T verificam as seguintes relações:

$$(1_1) \qquad (M J_h)_{ik} = (M J_k)_{ih};$$

$$(1_2) \qquad (J_h^T M)_{ik} = (J_i^T M)_{hk}$$

para toda a matriz quadrada M, de ordem n.

Tem lugar o

Teorema 1. *φ é linear, se e só se existem matrizes simétricas, regulares e constantes,* A, B, *tais que:*

$$(2) \qquad \mathrm{J^T A J = B}$$

a) Se φ é linear, J, J^T são constantes. Para toda a matriz A simétrica, regular e constante, a matriz $B = J^T A J$ é simétrica, regular e constante.

b) Reciprocamente, se se verifica a hipótese do Teo. 1, derive-se (2) em ordem a x_h:

$$J_h^T A J + J^T A J_h = 0$$

ou

$$J_h^T A J + (J_h^T A J)^T = 0.$$

Então:

$$(J_h^T A J)_{ik} + (J_h^T A J)_{ki} = 0, \qquad \forall h, i, k,$$

o que se pode escrever, por (1_2):

$$(J_h^T A J)_{ik} + (J_k^T A J)_{hi} = 0.$$

Ainda

$$(J_i^T A J)_{kh} + (J_h^T A J)_{ik} = 0$$

$$-(J_k^T A J)_{hi} - (J_i^T A J)_{kh} = 0.$$

[1] *Sempre que seja manifesto que uma letra representa um índice tomando valores em* $\{1, \cdot, n\}$ suprimiremos esta última indicação.

Somando estas 3 relações, vem:

$$2 (J_h^T A J)_{ik} = 0, \qquad \forall h, i, k,$$

donde

$$J_h^T A J = 0, \qquad \forall h.$$

Como A e J são matrizes regulares, resulta:

$$J_h^T = 0, \qquad \forall h.$$

J^T — e portanto J — é constante. φ é então linear.

A condição de simetria para as matrizes A e B, é essencial. Sem ela, não só a demonstração anterior deixa de ser válida, mas também o próprio Teo. 1 é falso. É o que se pode mostrar com os dois seguintes contra-exemplos.

1 — Seja a matriz:

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Para qualquer matriz quadrada M, de 2.ª ordem, tem-se:

$$M^T S M = (\det M) S.$$

Considere-se, então, no plano euclideano R^2, num homeomorfismo φ, de classe C^2, cujo determinante jacobiano seja constante. A sua matriz jacobiana J verifica (2), com $A = S$, $B = (\det M) S$. Com excepção da simetria, são satisfeitas as condições do Teo. 1. No entanto, φ não é necessariamente linear [1].

2 — No espaço das fases de um sistema material com n graus de liberdade, a condição necessária e suficiente para que um homeomorfismo de classe C^2 seja uma transformação canónica, é que exista um escalar constante μ, tal que $J^T S J = \mu S$, onde S é a matriz de ordem $2n$:

$$S = \begin{bmatrix} [0]_n & [E]_n \\ -[E]_n & [0]_n \end{bmatrix}$$

[1] É o caso, por ex., da aplicação $y_1 = \varphi_1(x_1, x_2) = x_1 + x_2^2$; $y_2 = \varphi_2(x_1, x_2) = x_1 + x_2^2 + a x_2$, com $a \neq 0$.

com $[0]_n$ e $[E]_n$, respectivamente matriz zero e matriz unidade de ordem n. Também neste caso são verificadas, com excepção da simetria, as condições do Teo. 1. No entanto as transformações canónicas não são necessariamente lineares.

2 — Grupo de invariância de uma forma diferencial quadrática

Seja $f = \sum_{i,k} a_{ik}\, dx_i\, dx_k$ uma forma diferencial quadrática em R^n. Dá-se o nome de grupo de invariância de f, ao conjunto I_f, das transformações de coordenadas generalizadas de R^n tais que $J^T A J = A$, onde $J(x)$ é a matriz jacobiana da transformação e $A(x)$ a matriz simétrica: $A = [a_{ik}]$. Diz-se também, para brevidade de linguagem, que I_f é o grupo de invariância da matriz A, representando-o então por I_A.

Desde que o domínio das transformações de I_A seja R^n, I_A é realmente um grupo de transformações. Além disso, como $(\det J)^2 \cdot \det A = \det A$, é $\det J = \pm 1$, se A é regular.

Do Teo. 1 resulta então:

Teorema 2. *O grupo de invariância de uma forma diferencial quadrática, regular e de coeficientes constantes, é um sub-grupo do grupo linear em* R^n.

Este teorema permite, por exemplo, concluir imediatamente que o grupo de Lorentz é linear. Com efeito, aquele grupo constitui o grupo de invariância da forma diferencial quadrática: $dx_1^2 + dx_2^2 + dx_3^2 - c^2\, dx_4^2$.

Na definição dada, consideraram-se as funções $\varphi(x)$ como representando transformações de coordenadas em R^n. Pode atribuir-se a estas funções o significado diferente de representarem transformações pontuais em R^n, sem que isso modifique a validade dos resultados obtidos. É o que faremos no seguimento deste trabalho.

3 — Transformações isométricas

Uma aplicação bijectiva φ de $X \subset R^n$ em R^n diz-se uma transformação isométrica — ou simplesmente uma isometria — se para todo $(x, x') \in X \times X$, é: distancia $[\varphi(x), \varphi(x')] =$ distancia $[x, x']$.

Tem lugar o seguinte:

Teorema 3. *Toda a isometria é uma transformação linear.*

Este teorema é válido para qualquer isometria, sem ser necessário fazer quaisquer hipóteses, quer relativas ao seu domínio X, quer de continuidade para φ e para as suas derivadas (embora, da definição, seja imediato que, pelo menos φ, é uma aplicação contínua). Vamos no entanto começar por fazer a sua demonstração admitindo que $X = R^n$, e que φ é de classe C^2. No seguimento abandonaremos estas restrições.

Tem-se então:

$$\forall\ (x, x') \in R^n \times R^n, \qquad |y' - y| = |x' - x|,$$

onde $y = \varphi(x)$, $y' = \varphi(x')$, e $|\mathbf{a}|$ representa o módulo do vector $\mathbf{a}$ do espaço vectorial E^n associado a R^n.

Utilizaremos a notação seguinte: sendo $\mathbf{u}$ e $\mathbf{v}$ dois vectores de E^n, o seu produto interno $(\mathbf{u}, \mathbf{v})$ será representado pela forma bilinear $G\mathbf{u}\cdot\mathbf{v}$, onde G é a matriz — constante — da métrica de R^n.

Pondo $\mathbf{u} = x' - x$, $\mathbf{v} = \varphi(x') - \varphi(x)$, deve ter-se então:

$$G\mathbf{u}\cdot\mathbf{u} = G\mathbf{v}\cdot\mathbf{v}.$$

Mas

$$\mathbf{v} = \varphi(x') - \varphi(x) = J(x)\mathbf{u} + \mathbf{0}(|\mathbf{u}|)$$

onde o vector $\mathbf{0}(|\mathbf{u}|)$ verifica:

$$\lim_{|\mathbf{u}|=0} \frac{\mathbf{0}(|\mathbf{u}|)}{|\mathbf{u}|} = \mathbf{0}, \qquad \mathbf{0}(|\mathbf{u}|) = \mathbf{0}, \quad \text{se } J \text{ é constante.}$$

Então:

$$G\mathbf{v}\cdot\mathbf{v} = G(J\mathbf{u} + \mathbf{0}(|\mathbf{u}|))\cdot(J\mathbf{u} + \mathbf{0}(|\mathbf{u}|)) = GJ\mathbf{u}\cdot J\mathbf{u} + 0(|\mathbf{u}|^2),$$

onde o escalar $0(|\mathbf{u}|^2)$ verifica

(3) $$\lim_{|\mathbf{u}|=0} \frac{0(|\mathbf{u}|^2)}{|\mathbf{u}|^2} = 0,$$

(4) $$0(|\mathbf{u}|^2) = 0, \quad \text{se } J \text{ é constante.}$$

A condição de isometria traduz-se então por:

$$\forall\ \mathbf{u} \in E^n \qquad G\,\mathbf{u}\cdot\mathbf{u} = G J\mathbf{u}\cdot J\mathbf{u} + 0(|\mathbf{u}|^2)$$

ou

$$\forall\ \mathbf{u} \in E^n \qquad G\,\mathbf{u}\cdot\mathbf{u} = J^T G J\mathbf{u}\cdot\mathbf{u} + 0(|\mathbf{u}|^2). \tag{5}$$

Vamos mostrar que (5) é equivalente a:

$$J^T G J = G. \tag{6}$$

Com efeito:

a) (6) implica (5). A relação (6) significa que $\varphi \in I_G$. Pelo Teo. 2, φ é linear e portanto J é constante. Então, por (4), e novamente por (6), (5) é verificado.

b) (5) implica (6). Tome-se em E^n um vector unitário $\mathbf{e}$. Pondo em (5) $\mathbf{u} = a\,\mathbf{e}$, com $a = \pm|\mathbf{u}|$, vem:

$$a^2 G\,\mathbf{e}\cdot\mathbf{e} = a^2 J^T G J\mathbf{e}\quad \mathbf{e} + 0(a^2)$$

ou

$$(G - J^T G J)\,\mathbf{e}\cdot\mathbf{e} = \frac{0(a^2)}{a^2}.$$

Como o primeiro membro desta relação é independente de a, será:

$$(G - J^T G J)\,\mathbf{e}\cdot\mathbf{e} = \lim \frac{0(a^2)}{a^2} = 0.$$

O anulamento da forma quadrática $(G - J^T G J)\,\mathbf{u}\cdot\mathbf{u}$, para todo o vector $\mathbf{u}$ de E^n, é, dada a simetria da matriz $G - J^T G J$, equivalente a (6).

Demonstrado o Teo. 3 nas hipóteses atrás formuladas, vamos mostrar que ele é válido ainda quando se abandonam aquelas hipóteses. Para isso teremos de utilizar dois teoremas de que nos vamos, por esta razão, ocupar.

Teorema 4. *O número de dimensões dos conjuntos de* R^n (1), *é invariante para as isometrias.*

Seja φ uma isometria, seja $X \subset R^n$ um conjunto de dimensão h e seja h_1 o número de dimensões de $\varphi(X)$. É então possível tomar, em X, $h+1$ pontos $x_0, x, \cdots, x_h$ tais que os vectores $\mathbf{u}_k = x_k - x_0$, $(k = 1, \cdots, h)$ sejam linearmente independentes. Escreva-se $\mathbf{v}_k = \varphi(x_k) - \varphi(x_0)$, $(k = 1, \cdots, h)$, e seja $\mathbf{v} = \sum_{k=1}^{h} \lambda_k \mathbf{v}_k$. É:

$$|\mathbf{v}|^2 = G\,\mathbf{v}\cdot\mathbf{v} = G(\Sigma\,\lambda_k \mathbf{v}_k)\cdot(\Sigma\,\lambda_i \mathbf{v}_i) = \sum_{ik} \lambda_i \lambda_k\, G\,\mathbf{v}_k\cdot\mathbf{v}_i =$$
$$= \sum_{ik} \lambda_i \lambda_k\, G\,\mathbf{u}_k\cdot\mathbf{u}_i = |\Sigma\,\lambda_k \mathbf{u}_k|^2.$$

Então $\mathbf{v} = \mathbf{0}$ implica $\lambda_1 = \cdots = \lambda_h = 0$. Os h vectores $\mathbf{v}_k$ são linearmente independentes, e portanto $h_1 \geqq h$. Considerando agora a isometria φ^{-1}, como $X = \varphi^{-1}(\varphi(x))$, conclue-se analogamente que $h \geqq h_1$, donde $h = h_1$.

(Na demonstração anterior utilisou-se o facto de ser $G\,\mathbf{v}_k\cdot\mathbf{v}_i = G\,\mathbf{u}_k\cdot\mathbf{u}_i$. Isto resulta de φ conservar as distâncias de x_0, x_i; x_0, x_k; e x_i, x_k).

Teorema 5. *Uma isometria de domínio* R^n *fica univocamente determinada pelos seus valores em* $n+1$ *pontos constituindo um conjunto de dimensão* n.

Sejam $x_0, x_1, \cdots, x_n$ os pontos indicados no enunciado, $y_k = \varphi(x_k)$, $(k = 0, 1, \cdots, n)$, as suas imagens por φ. Ponha-se $\mathbf{u}_k = x_k - x_0$, $\mathbf{v}_k = y_k - y_0$, $(k = 1, \cdots, n)$. Os n vectores $\mathbf{u}_k$ são linearmente independentes, e, por consequência, os n vectores $\mathbf{v}_k$ também o são (como resulta da demonstração do Teo. 4). Seja φ_1 uma outra isometria, de domínio R^n, tal que $\varphi_1(x_k) = y_k$, $(k = 0, 1, \cdots, n)$. Para todo $x \in R^n$ sejam: $y = \varphi(x)$, $z = \varphi_1(x)$, $\mathbf{v} = y - y_0$, $\mathbf{w} = z - y_0$. Tem-se:

$$|\mathbf{v} - \mathbf{v}_k| = |\mathbf{w} - \mathbf{v}_k|, \qquad \forall\, k,$$

pois qualquer destes valores é igual à distância de x_k a x.

(1) Designamos por número de dimensões de um conjunto $X \subset R^n$, o infimo do número de dimensões dos sub-espaços de R^n que contêm X, ou, equivalentemente, o número de dimensões do sub-espaço de R^n gerado por X.

Então:

$$G(\mathbf{v}-\mathbf{v}_k)\cdot(\mathbf{v}-\mathbf{v}_k)=G(\mathbf{w}-\mathbf{v}_k)\cdot(\mathbf{w}-\mathbf{v}_k)$$

ou seja:

$$|\mathbf{v}|^2-2G\mathbf{v}\cdot\mathbf{v}_k+|\mathbf{v}_k|^2=|\mathbf{w}|^2-2G\mathbf{w}\cdot\mathbf{v}_k+|\mathbf{v}_k|^2.$$

Como

$$|\mathbf{v}|^2=|\mathbf{w}|^2=[\text{distância } (x_0,x)]^2$$

resulta

$$G(\mathbf{v}-\mathbf{w})\cdot\mathbf{v}_k=0, \qquad \forall k,$$

e como os vectores $\mathbf{v}_1,\cdots,\mathbf{v}_n$ são linearmente independentes:

$$\mathbf{v}=\mathbf{w},$$

donde

$$\varphi(x)=\varphi_1(x), \qquad \forall x\in R^n.$$

Este teorema vai-nos permitir prescindir da hipótese de ser φ de classe C^2, para o Teo. 3. Seja φ uma isometria de domínio R^n. Tomem-se $n+1$ pontos $x_0,x_1,\cdots,x_n$, formando um conjunto de dimensão n. Usando as mesmas notações da demonstração do Teo. 5, defina-se uma matriz J pelas condições:

$$J\mathbf{u}_k=\mathbf{v}_k, \qquad (k=1,\cdots,n).$$

J fica univocamente determinado por estas relações e verifica a igualdade $J^TGJ=G$.

Seja φ_1 a isometria, de domínio R^n:

$$\forall\ x\in R^n: \qquad \varphi_1(x)=y_0+J(x-x_0).$$

É $\varphi_1(x_0)=y_0$, $\varphi_1(x_k)=y_k$, $(k=1,\cdots,n)$. Então, pelo Teo. 5, é $\varphi=\varphi_1$. Portanto φ é linear.

Em virtude da sua linearidade, cada isometria φ de domínio R^n, induz em E^n uma aplicação linear — que representamos pelo mesmo símbolo —:

$$\forall\ \mathbf{u}\in E^n, \qquad \varphi(\mathbf{u})=\varphi(x)-\varphi(x_0), \qquad \text{se } \mathbf{u}=x-x_0,$$

ou seja:

$$\varphi(\mathbf{u})=J\mathbf{u},$$

onde J é a matriz de φ. Esta aplicação é uma isometria vectorial:

$$|\varphi(\mathbf{u})|=|\mathbf{u}| \qquad (\varphi(\mathbf{u}),\ \varphi(\mathbf{v}))=(\mathbf{u},\mathbf{v}).$$

Resta mostrar que o Teo. 3, é ainda válido para as isometrias cujo domínio X é um subconjunto próprio de R^n. Para isso vai-se provar que uma tal isometria φ coincide com a restrição a X de uma ou mais isometrias de domínio R^n. A linearidade destas últimas implica assim a de φ.

Seja $h\leq n$ o número de dimensões de X. Se $h=n$, a construção anterior e a demonstração do Teo. 5 mostram que existe uma, e só uma, isometria que prolonga φ a R^n. Se $h<n$, tomem-se, em X, $h+1$ pontos $x_0,x_1,\cdots,x_h$, formando um conjunto com h dimensões. Sejam $y_k=\varphi(x_k)$, $(k=0,1,\cdots,h)$, $\mathbf{u}_k=x_k-x_0$, $\mathbf{v}_k=y_k-y_0$. Tomem-se dois sistemas ortonormados, de $n-h$ vectores $(\mathbf{u}_{h+1},\cdots,\mathbf{u}_n)$, $(\mathbf{v}_{h+1},\cdots,\mathbf{v}_n)$, o primeiro ortogonal aos $\mathbf{u}_k$, $(k=1,\cdots,h)$, o segundo ortogonal aos $\mathbf{v}_k$, $(k=1,\cdots,h)$.

A aplicação

$$\varphi'(x_k)=y_0+\mathbf{v}_k, \qquad (k=1,\cdots,n)$$

onde $x_k=x_0+\mathbf{u}_k$ é uma isometria de domínio $X'=\{x_0,x_1,\cdots,x_n\}$, a qual, como se viu, é prolongavel a R^n por uma única isometria φ'_1. Finalmente, a restrição de φ'_1 a X, coincide com φ. A demonstração deste último facto é perfeitamente análoga à que foi utilizada para demonstrar o Teo. 5, tendo em atenção que $\varphi(X)$ e $\varphi'_1(X)$ têm ambos dimensão h. Fica deste modo completamente demonstrado o Teo. 3.

4 — Sinal de uma isometria de domínio R^n. Prolongamento a R^n de isometrias cujo domínio é um subconjunto próprio de R^n

Como as isometrias de domínio R^n constituem o grupo de invariância da matriz da métrica G, o seu determinante é igual a ± 1. Diz-se que uma tal isometria é positiva — ou própria — se $\det J=+1$, negativa — ou imprópria — se $\det J=-1$. Além disso, como a matriz J representa uma isometria vectorial em E^n — e também no espaço vectorial complexo C^n —, será, para qualquer valor próprio λ de J, e correspondente vector próprio $\mathbf{z}$:

$$|J\mathbf{z}| = |\lambda \mathbf{z}| = |\lambda|\,|\mathbf{z}| = |\mathbf{z}|,$$

donde resulta: $|\lambda| = 1$.

Vimos na parte final do número anterior que toda a isometria de domínio $X \neq R^n$, pode ser prolongada a R^n por uma outra isometria. Este prolongamento, em geral, não é único, como aliás o deixa prever a construção ali feita. Vamos examinar aqui esta questão com mais detalhe, em especial quando o número de dimensões de X é igual a n, a $n-1$ e a $n-2$, já que são estes os casos que maior interesse apresentam nas aplicações à Cinemática do Sólido invariável.

A determinação dos prolongamentos a R^n de qualquer isometria de domínio X pode ser reduzida à de todos os prolongamentos a R^n (por isometrias), da aplicação idêntica de X sobre X. Com efeito, seja I_G o conjunto de todas as isometrias de domínio R^n (I_G coincide com o grupo de invariância da matriz G). Para cada $\varphi_1 \in I_G$, seja F_{φ_1} a aplicação bijectiva de I_G sobre I_G definida por:

$$\forall \varphi' \in I_G \qquad F_{\varphi_1}(\varphi') = \varphi_1 \circ \varphi'.$$

Se φ é uma isometria de domínio X, e φ_1 um prolongamento de φ a R^n, φ' será também um prolongamento de φ a R^n se e só se φ' for a imagem por F_{φ_1} de uma isometria de domínio R^n, cuja restrição a X seja a aplicação idêntica de X sobre X. Designemos por I_X o conjunto destas últimas isometrias, por E_X o sub-espaço vectorial gerado por X[1], e por E'_X o subespaço de E^n ortogonal a E_X. Então:

$$(7) \qquad \forall \varphi_X \in I_X \begin{cases} J_X \mathbf{u} = \mathbf{u}, & \forall \mathbf{u} \in E_X \\ J_X \mathbf{u}' \in E'_X, & \forall \mathbf{u}' \in E'_X \end{cases}.$$

Estas relações mostram que J_X tem, pelo menos, h valores próprios iguais a $+1$, e h vectores próprios linearmente indepen-

(1) Mais correctamente, E_X é o sub-espaço vectorial gerado pelos vectores definidos por todos os pares de pontos de X.

dentes associados àqueles valores próprios. O produto dos seus restantes valores próprios — todos de módulo igual a 1 — será igual a $+1$, se φ_X for próprio, e igual a -1, se φ_X for impróprio. Desta observação decorre, em particular, que, se n e h têm paridades diferentes, para cada isometria própria. φ_X pertencente a I_X existe um subconjunto Y de R^n com, pelo menos, $h+1$ dimensões, tal que $\varphi_X \in I_Y$. Se h e n têm a mesma paridade, o mesmo acontece para cada isometria imprópria de I_X.

Como E^n é a soma directa de E_X e E'_X, a cada isometria vectorial θ em E'_X, corresponde uma isometria de I_X definida por:

$$\varphi_X(x) = x_0 + \mathbf{u} + \theta(\mathbf{u}'),$$

onde $x_0 \in X$, e $\mathbf{u} + \mathbf{u}'$ é a decomposição do vector $x - x_0$ segundo E_X e E'_X. Esta correspondência é bijectiva.

a) *Caso em que* X *tem* n *dimensões.* Neste caso, $E_X = E^n$, $E'_X = \{\mathbf{0}\}$. De (7) resulta que $J_X = E$. O conjunto I_X é composto apenas pela aplicação idêntica de R^n sobre R^n. Portanto, qualquer isometria de domínio X admite um único prolongamento a R^n. Este resultado é equivalente ao Teo. 5, juntamente com a possibilidade de prolongar qualquer isometria a R^n.

b) *Caso em que* X *tem* n—1 *dimensões.* Neste caso E'_X tem uma dimensão, i. e.:

$$\mathbf{u}' \neq \mathbf{0} \qquad \mathbf{u}' \in E'_X \qquad \mathbf{v}' \in E'_X \Rightarrow \mathbf{v}' = \lambda \mathbf{u}'.$$

Existem portanto apenas duas isometrias vectoriais em E'_X:

$$\theta_1(\mathbf{u}') = \mathbf{u}', \qquad \theta_2(\mathbf{u}') = -\mathbf{u}', \qquad \forall \mathbf{u}' \in E'_X,$$

e portanto I_X é composto apenas pelas duas isometrias:

$$\forall x \in R^n \begin{cases} \varphi_X(x) = x_0 + \mathbf{u} + \mathbf{u}' = x & \text{(isometria idêntica)} \\ \varphi'_X(x) = x_0 + \mathbf{u} - \mathbf{u}' & \text{(isometria negativa),} \end{cases}$$

onde $x_0 \in X$ e $\mathbf{u} + \mathbf{u}'$ é a decomposição de $x - x_0$ segundo E_X e E'_X.

Podemos então enunciar os dois seguintes teoremas.

TEOREMA 6. *Dada uma isometria* φ, *de domínio* X *com* n — 1 *dimensões, existem duas e só duas, isometrias distintas que prolongam* φ *a* R^n. *Estas isometrias têm sinais contrários.*

TEOREMA 7. *Uma isometria de domínio* R^n *fica univocamente determinada pelo seu sinal e pelos valores que toma em* n *pontos formando um conjunto de dimensões* n — 1.

c) *Caso em que* X *tem* n — 2 *dimensões.* Neste caso E'_X tem 2 dimensões. Tomando em E'_X uma base ortonormada $\mathbf{e}'_1, \mathbf{e}'_2$, obtém-se todas as isometrias vectoriais em E'_X, por

$$(8) \qquad \begin{cases} \theta_1(\mathbf{e}'_1) = \lambda_1 \mathbf{e}'_1 + \lambda_2 \mathbf{e}'_2 \\ \theta_1(\mathbf{e}'_2) = \mu_1 \mathbf{e}'_1 + \mu_2 \mathbf{e}'_2 , \end{cases}$$

dando a $\lambda_1, \lambda_2, \mu_1, \mu_2$ todos os valores reais compatíveis com as condições de isometria. Estas são:

$$(9) \qquad \lambda_1^2 + \lambda_2^2 = \mu_1^2 + \mu_2^2 = 1$$

$$(10) \qquad \lambda_1 \mu_1 + \lambda_2 \mu_2 = 0 .$$

É ainda

$$(11) \qquad \lambda_1 \mu_2 - \lambda_2 \mu_1 = \det J_X ,$$

onde J_X é a matriz de isometria φ_X correspondente a θ.

As condições (9) ficam automaticamente satisfeitas, tomando dois parâmetros reais α e β, pondo:

$$(12) \qquad \lambda_1 = \cos\alpha , \quad \lambda_2 = \operatorname{sen}\alpha , \quad \mu_1 = \cos\beta , \quad \mu_2 = \operatorname{sen}\beta ;$$

(10) impõe a α e β a relação:

$$(10') \qquad \cos(\beta - \alpha) = 0$$

e, de (11),

$$(11') \qquad \operatorname{sen}(\beta - \alpha) = \det J_X .$$

As isometrias vectoriais em E'_X obtidas por (8) e (12) com $\beta = \alpha + \pi/2$ dão origem a isometrias φ_X positivas. As que se obtém com $\beta = \frac{3}{2}\pi + \alpha$ dão origem a isometrias φ_X negativas.

c′) *Isometrias positivas.* Tem-se

$$\lambda_1 = \cos\alpha , \qquad \lambda_2 = \operatorname{sen}\alpha , \qquad \mu_1 = -\operatorname{sen}\alpha , \qquad \mu_2 = \cos\alpha .$$

Obtém-se uma aplicação bijectiva do intervalo $[0, 2\pi[$ sobre I_X^+, representada por:

$$\varphi_{X,\alpha}(x) = x_0 + J_{X,\alpha}(x - x_0) , \qquad x_0 \in X ,$$

onde a matriz $J_{X,\alpha}$ é definida por:

$$J_{X,\alpha}\mathbf{u} = \mathbf{u} , \qquad \forall\, \mathbf{u} \in E_X ;$$

$$J_{X,\alpha}\mathbf{e}_1^1 = \cos\alpha\, \mathbf{e}_1^1 + \operatorname{sen}\alpha\, \mathbf{e}_2^1 ;$$

$$J_{X,\alpha}\mathbf{e}_2^1 = -\operatorname{sen}\alpha\, \mathbf{e}_1^1 + \cos\alpha\, \mathbf{e}_2^1 .$$

Para $\alpha = 0$, é $J_{X,0} = E$. Portanto $\varphi_{X,0}$ é a aplicação idêntica de R^n sobre R^n.

Pondo $\mathbf{z}_1^1 = \mathbf{e}_1^1 - i\,\mathbf{e}_2^1$ e portanto $\mathbf{z}_1^{1*} = \mathbf{z}_2^1 = \mathbf{e}_1^1 + i\,\mathbf{e}_2^1$ vem:

$$J_{X,\alpha}\mathbf{z}_1^1 = e^{i\alpha}\mathbf{z}_1^1 , \qquad J_{X,\alpha}\mathbf{z}_2^1 = e^{-i\alpha}\mathbf{z}_2^1 ,$$

o que mostra serem os dois últimos valores próprios de $J_{X,\alpha}$ iguais a $e^{i\alpha}, e^{-i\alpha}$, reduzindo-se a $+1, +1$ para $J_{X,0}$, e a $-1, -1$ para $J_{X,\pi}$.

Note-se que, no caso c′), os dois últimos valores próprios de $J_{X,\alpha}$ variam efectivamente com α, enquanto que os correspondentes vectores próprios são independentes de α (com a restrição dos casos $\alpha = 0, \alpha = \pi$, nos quais todos os vectores de E'_X e todas as suas combinações lineares de coeficientes complexos são vectores próprios das matrizes correspondentes).

c″) *Isometrias negativas.* Neste caso é $\lambda_1 = \cos\alpha$, $\lambda_2 = \operatorname{sen}\alpha$, $\mu_1 = \operatorname{sen}\alpha$, $\mu_2 = -\cos\alpha$. Obtém-se ainda uma aplicação bijectiva do mesmo intervalo $[0, 2\pi[$ sobre I_X^-, representada por:

$$\varphi'_{X,\alpha} = x_0 + J'_{X,\alpha}(x - x_0) , \qquad x_0 \in X ,$$

com $J'_{X,\alpha}$ definida por:

$$J'_{X,\alpha}\,\mathbf{u}=\mathbf{u}, \qquad \forall\, \mathbf{u}\in E_X;$$

$$J'_{X,\alpha}\,\mathbf{e}'_1=\cos\alpha\,\mathbf{e}'_1+\operatorname{sen}\alpha\,\mathbf{e}'_2;$$

$$J'_{X,\alpha}\,\mathbf{e}'_2=\operatorname{sen}\alpha\,\mathbf{e}'_1-\cos\alpha\,\mathbf{e}'_2.$$

Pondo:

$$\mathbf{u}'_1=\cos\alpha/2\,\mathbf{e}'_1+\operatorname{sen}\alpha/2\,\mathbf{e}'_2, \qquad \mathbf{u}'_2=\operatorname{sen}\alpha/2\,\mathbf{e}'_1-\cos\alpha/2\,\mathbf{e}'_2,$$

tem-se

$$J'_{X,\alpha}\,\mathbf{u}'_1=\mathbf{u}'_1, \qquad J'_{X,\alpha}\,\mathbf{u}'_2=-\,\mathbf{u}'_2$$

o que mostra terem os dois últimos valores próprios de $J'_{X,\alpha}$ os valores $+1$ e -1. Para cada α, $\varphi'_{X,\alpha}$ pertence a I_{Y_α}, onde Y_α é o sub-espaço de R^n gerado por um ponto x_0 de X e pelos vectores pertencentes à soma directa de E_X com

$$\{\mathbf{u}'_1=\cos\alpha/2\,\mathbf{e}'_1+\operatorname{sen}\alpha/2\,\mathbf{e}'_2\}.$$

Note-se finalmente que no caso $c'')$ tem lugar a situação oposta à que se assinalou no caso $c')$: os dois últimos valores próprios de $J'_{X,\alpha}$ são independentes de α, enquanto que os correspondentes vectores próprios variam efectivamente com α.

Os resultados obtidos em $c)$ justificam o seguinte teorema:

Teorema 8. *Uma isometria φ, cujo domínio tem $n-2$ dimensões, pode ser prolongada a R^n por uma família infinita de isometrias positivas distintas e por uma família infinita de isometrias negativas distintas. Cada uma destas famílias é uni-paramétrica, podendo ser representada por um parâmetro do domínio $[0, 2\pi[$.*

B. APLICAÇÃO À CINEMÁTICA DO SÓLIDO INVARIÁVEL

5 — Sistemas materiais. Deslocamentos de sistemas materiais

Limitando-nos apenas ao ponto de vista cinemático, pode definir-se sistema material como um conjunto S, ao qual se associa um conjunto C de aplicações biunívocas de S em R^3 tal que a imagem de

S por qualquer destas aplicações é um conjunto fechado e limitado de R^3 (1).

Se $\psi\in C$, ψ diz-se uma representação geométrica de S. A $\psi(S)\subset R^3$ dá-se o nome de imagem geométrica, ou configuração, de S, na representação geométrica ψ. Se $P\in S$, $\psi(P)\in\psi(S)=X$ diz-se posição do ponto P, de S, na configuração X.

Se $\psi_1, \psi_2\in C$, $X_1=\psi_1(S)$, $X_2=\psi_2(S)$, a aplicação $\varphi=\psi_2\circ\psi_1^{-1}$ é uma bijecção de X_1 sobre X_2, à qual se dá o nome de deslocamento do sistema material S, da configuração X_1 — configuração inicial —, para a configuração X_2 — configuração final —. Se $P\in S$, ao vector $\psi_2(P)-\psi_1(P)$ dá-se o nome de vector deslocamento do ponto P de S no deslocamento $\varphi=\psi_2\circ\psi_1^{-1}$. Estas aplicações devem satisfazer a condição de serem homeomorfismos.

De certo modo, pode dizer-se que o que caracteriza a natureza de um determinado sistema material, são as condições a que estão sujeitos os deslocamentos que pode experimentar, e que caracterizam a família F constituída por estes.

Assim, o sistema material S diz-se um sólido invariável, se a família F é composta apenas por isometrias positivas. Dos resultados obtidos em A, decorre que os deslocamentos dos sólidos invariáveis são aplicações lineares de domínio contido em R^3.

Um deslocamento φ de um sólido invariável S fica definido pelo seu domínio X — configuração inicial — pelas posições inicial e final, x_0, y_0, de um seu ponto e pela matriz J da isometria. Esta, conforme a discussão feita no n.º 4, só é univocamente determinada se o número de dimensões de X for superior a 1.

Se $\varphi:(X, x_0, y_0, J)$, tem se:

$$\forall\, x\in X:\ \varphi(x)=y_0+J(x-x_0).$$

A matriz J pertence ao grupo de invariância da matriz G, da métrica de R^3 determinada pelo referencial tomado neste espaço. Se aquele referencial for orto-normado, J pertence ao grupo ortogonal $(J\,J^T=E)$.

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(1) Além de condições de continuidade que adiante se indicarão, a utilização deste conceito em cada caso concreto impõe outras condições a C. Assim, e além do exemplo fornecido pelo caso de S ser um sólido invariável de que nos vamos ocupar, se, por exemplo, o sistema material S estiver sujeito a ligações holónomas bilaterais que lhe conferem um número finito, k, de graus de liberdade, existe uma bijecção entre C e um paralelipípedo de R^k.

6 — Número de dimensões de um sólido invariável

Do Teo. 4 conclue-se que todas as configurações de um sólido invariável têm o mesmo número de dimensões. Isto justifica o uso dos termos: sólido tri-dimensional, sólido bi-dimensional (ou plano), sólido uni-dimensional (ou rectilíneo).

O Teo. 5 permite enunciar o teorema seguinte:

Teorema 9. *Um deslocamento de um sólido tri-dimensional fica completamente representado pelos deslocamentos de 4 dos seus pontos não complanares. Estes deslocamentos devem obedecer à condição de constituirem uma isometria positiva.*

Os Teo. 6 e Teo. 7, permitem enunciar o seguinte teorema:

Teorema 10. *Um deslocamento de um sólido tri-dimensional, ou bi-dimensional, fica completamente representado pelos deslocamentos de 3 dos seus pontos não colineares. Estes deslocamentos devem obedecer à condição de constituirem uma isometria.*

Para o caso dos sólidos rectilíneos, é evidente que um seu deslocamento fica completamente representado pelos deslocamentos de 2 dos seus pontos (os quais devem obedecer à condição de constituirem uma isometria). No entanto, o prolongamento a R^3 dos deslocamentos destes sólidos, é, como vimos, indeterminado, o mesmo acontecendo às matrizes que os representam. Por esta razão, estes sólidos constituem um caso singular, para o qual não são aplicáveis alguns dos resultados que adiante se indicarão.

7 — Espaço solidário com um sólido invariável

A condição de que toda a configuração de um sistema material é um conjunto limitado (imposta pela natureza dos corpos reais, a cujo comportamento vai ser aplicado aquele conceito) implica que seja sempre um subconjunto próprio de R^3. Este facto dá origem a algumas dificuldades, que se podem evitar, no caso dos sólidos invariaveis, introduzindo uma noção que, empregando uma linguagem bastante expressiva, se designa muitas vezes, por espaço solidário com o sólido. Consiste em associar a cada configuração do sólido um espaço

R^3, de forma tal que os espaços R^3 correspondentes às configurações X_1 e X_2 se apliquem um sobre o outro pela isometria positiva que prolonga a R^3 o deslocamento do sólido de X_1 para X_2. Assim, se x é um ponto qualquer de R^3, e o considerarmos como a posição de um ponto do espaço solidário com o sólido numa configuração X_1, a posição do mesmo ponto na configuração X_2 será: $y = \theta(x)$, onde θ é a isometria positiva que prolonga a R^3 o deslocamento de X_1 para X_2.

Daqui em diante, suporemos sempre que os domínios dos deslocamentos de um sólido invariável são espaços R^3, com a reserva de que, no caso dos sólidos rectilíneos, aquelas extensões não são completamente determinadas. Sendo φ um deslocamento de um sólido invariável, fica completamente indicado pelas posições inicial e final de um seu ponto e pela sua matriz. Se $\varphi(x_0, y_0, J)$, é,

$$\forall\, x \in R^3, \qquad \varphi(x) = y_0 + J(x - x_0).$$

8 — Composição de deslocamentos

Dados os deslocamentos $\varphi_1(x_0, y_0, J_1)$ e $\varphi_2(x'_0, y'_0, J_2)$, o deslocamento $\varphi = \varphi_2 \circ \varphi_1$ diz-se o deslocamento composto de φ_1 e φ_2. É portanto o deslocamento que se obtém efectuando sucessivamente os deslocamentos φ_1 e φ_2. Tem-se:

$$(13) \quad \forall\, x \in R^3, \quad \varphi(x) = \varphi_2(\varphi_1(x)) = \varphi_2(y_0 + J_1(x - x_0)) = \\ = \varphi_2(y_0) + J_2 J_1(x - x_0) = z_0 + J_2 J_1(x - x_0),$$

onde

$$z_0 = (\varphi_2 \circ \varphi_1)(x_0).$$

Em relação a esta operação, a família F dos deslocamentos de um sólido invariável é um subgrupo do grupo das isometrias positivas de R^3. Com efeito:

a) Se $\varphi_1, \varphi_2 \in F$, então $\varphi_2 \circ \varphi_1 \in F$.

Esta propriedade decorre da hipótese que se pode enunciar do seguinte modo: se $\varphi \in F$, e X é uma configuração do sólido, $\varphi(X)$ é também uma configuração do sólido.

b) Se $\varphi \in F$, então $\varphi^{-1} \in F$.

c) O deslocamento nulo — ou aplicação idêntica — pertence a F.

Se F coincide com o grupo das isometrias positivas de R^3, o sólido diz-se livre. Neste caso, o seu número de graus de liberdade é 6, pois existe uma bijecção entre F e o paralelipípedo:

$$-\infty < q_1 < +\infty, \quad -\infty < q_2 < +\infty, \quad -\infty < q_3 < +\infty,$$
$$0 \leq q_4 < 2\pi, \quad 0 \leq q_5 < 2\pi, \quad 0 \leq q_6 < 2\pi,$$

onde q_1, q_2, q_3 representam as componentes do vector deslocamento de um ponto do sólido, q_4, q_5, q_6 os ângulos de Euler que definem a matriz J.

Se F é subgrupo próprio do grupo das isometrias positivas de R^3, diz-se que o sólido está sujeito a ligações. Neste caso o seu número de graus de liberdade é inferior a 6.

9 — Alguns deslocamentos particulares

a) *Translações.* Um deslocamento φ, de um sólido invariável S, diz-se uma translação, se todos os pontos do sólido (e do espaço solidário com ele) têm o mesmo vector deslocamento. A este vector dá-se o nome de vector da translação de φ.

$\varphi(x_0, y_0, J)$ será então uma translação, se e só se:

$$\forall\, x \in R^3, \qquad \varphi(x) - x = y_0 - x_0;$$

ou

$$\forall\, x \in R^3, \qquad y_0 + J(x - x_0) - x = y_0 - x_0;$$

ou

$$\forall\, x \in R^3, \qquad J(x - x_0) = x - x_0.$$

φ é então uma translação, se e só se $J = E$.

O vector da translação é: $\bar{\mathbf{u}} = y_0 - x_0$ e tem-se:

$$\forall\, x \in R^3, \qquad \varphi(x) = x + \bar{\mathbf{u}}.$$

Se φ_1 e φ_2 são duas translações, de vectores $\bar{\mathbf{u}}_1$ e $\bar{\mathbf{u}}_2$ respectivamente, a expressão (13) mostra que a sua composição é ainda uma translação. Por outro lado $(\varphi_2 \circ \varphi_1)(x) = \varphi_2(x + \bar{\mathbf{u}}_1) = x + \bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 = x + \bar{\mathbf{u}}_2 + \bar{\mathbf{u}}_1 = \varphi_1\,(x + \bar{\mathbf{u}}_2) = (\varphi_1 \circ \varphi_2)(x)$.

Tem-se então o

Teorema 11. *O conjunto* T *de todas as translações de um sólido invariável é um subgrupo comutativo do grupo das isometrias positivas de* R^3.

b) *Rotações.* Um deslocamento φ de um sólido invariável S diz-se uma rotação em torno de um ponto x_0, se este ponto permanece fixo nesse deslocamento (1).

Na rotação $\varphi : (x_0, x_0, J)$ tem-se:

$$\forall\, x \in R^3, \qquad \varphi(x) = x_0 + J(x - x_0).$$

E de verificação imediata que o deslocamento composto de duas rotações em torno de um mesmo ponto é ainda uma rotação em torno desse ponto, cuja matriz é o produto das matrizes das rotações componentes. Como estas últimas matrizes, em geral, não comutam uma com a outra tem lugar o:

Teorema 12. *O conjunto* R *das rotações em torno de um mesmo ponto é um sub-grupo não comutativo do grupo das isometrias positivas.*

Seja $\varphi(x_0, x_0, J)$ uma rotação em torno de x_0. Neste deslocamento, x fica também fixo, se:

$$\varphi(x) = x = x_0 + J(x - x_0).$$

Pondo $x - x_0 = \mathbf{u}$, tem-se então $J\mathbf{u} = \mathbf{u}$.

Então, na rotação φ permanecem fixos, todos e só os pontos x tais que $x = x_0 + \mathbf{u}$, onde $\mathbf{u}$ seja vector próprio da matriz J correspondente ao valor próprio $+1$. Vamos mostrar que, com excepção do deslocamento nulo $\varphi_0(x_0, x_0, E)$, existe uma e só uma direcção de vectores reais naquelas condições. Este resultado traduz a propriedade de que numa rotação em torno de x_0, ficam fixos todos e só

(1) Esta definição só é válida por se admitirem apenas deslocamentos que são isometrias positivas. Se se admitissem também isometrias negativas, o deslocamento $\varphi : (x_0, x_0, J)$ seria uma rotação em torno de x_0, se $\det J = +1$; uma reflexão relativa a x_0, se $J = -E$; e um deslocamento composto de uma rotação e uma reflexão, se $\det J = -1$, com $J \neq -E$.

os pontos da recta $Y(x_0, \mathbf{u})$, à qual se dá o nome de eixo da rotação considerada [1].

Com efeito, se $\lambda_1, \lambda_2, \lambda_3$ forem os valores próprios da matriz J

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = +1 = \lambda_1 \lambda_2 \lambda_3 .$$

Por outro lado, se um destes valores for um complexo c, um outro destes valores será o complexo conjugado c^*, de c — e portanto o terceiro λ será igual a $+1$, pois $c c^* = +1$ —. Então são apenas possíveis as seguintes combinações para aqueles valores:

$$(+1,\ +1,\ +1) \quad (+1,\ -1,\ -1) \quad (+1,\ c,\ c^*).$$

O primeiro caso corresponde ao deslocamento nulo, já que uma matriz isométrica de R^3 admite como vectores próprios 3 vectores do espaço vectorial complexo C^3 linearmente independentes. Os outros dois casos justificam a propriedade indicada.

Se φ_1 e φ_2 são duas rotações em torno do eixo $X(x_0, \mathbf{u})$, o deslocamento composto de φ_1 e φ_2 é também uma rotação de eixo Y, já que $J_2 J_1 \mathbf{u} = J_2 \mathbf{u} = \mathbf{u}$. Além disso, como veremos, φ_1 e φ_2 comutam entre si, isto é: $\varphi_2 \circ \varphi_1 = \varphi_1 \circ \varphi_2$. Tem então lugar o teorema:

Teorema 13. *O conjunto de todas as rotações em torno de um mesmo eixo é um sub-grupo comutativo do grupo das rotações em torno de um ponto desse eixo e portanto também do grupo das isometrias positivas.*

Dá-se o nome de amplitude angular de uma rotação φ em torno de um eixo, $Y(x_0, \mathbf{u})$, ao ângulo de que roda um qualquer vector normal ao eixo.

Como $\varphi \in I_Y^+$ (no sentido dado a este símbolo no n.º 4), pode pôr-se, com a notação usada em c') daquele número: $J = J_{Y,\alpha}$, como se verifica imediatamente.

Então, se φ_1 e φ_2 são duas rotações em torno de um eixo Y, de amplitudes angulares α, β, respectivamente, tem-se $J_1 = J_{Y,\alpha}$, $J_2 = J_{Y,\beta}$ e $J_1 \cdot J_2 = J_{Y,\alpha+\beta} = J_2 J_1$, o que justifica a afirmação feita atrás.

(1) Este resultado é conhecido pelo nome de Teorema de Euler relativo às rotações de um sólido invariável.

Com o estudo deste caso, pode apreciar-se melhor, a indeterminação ligada à noção de espaço solidário com um sólido, se este é rectilíneo. Com efeito, para uma dada configuração deste, podem associar-se espaços R^3 diferentes, que se obtenham uns dos outros por qualquer rotação em torno daquela configuração.

c) Decomposições do deslocamento de um sólido em pares formados por uma translação a uma rotação. Vamos mostrar que todo o deslocamento $\varphi(x_0, y_0, J)$, de um sólido invariável, se pode decompor numa translação, cujo vector translação é o vector deslocamento de um ponto, segundo φ, e numa rotação em torno da posição final desse ponto, de matriz J.

Seja x_0, o ponto considerado, e $\varphi(x_0, y_0, J)$. Considerem-se a translação $T(\mathbf{u} = \varphi(x_0) - x_0)$, e a rotação $R(y_0, y_0, J)$. Tem-se:

$$R\,T(x_0, R\,T(x_0), \dot{J}E) = J_1(x_0, y_0, J) = \varphi .$$

Invertendo a ordem dos deslocamentos componentes, pode também decompor-se φ, numa rotação, em torno de um ponto qualquer, de matriz J, e numa translação cujo vector é o vector deslocamento daquele ponto no deslocamento φ. Considere-se a rotação $R_1(x_0, x_0, J)$ e $T_1(\mathbf{u}_1 = \varphi(x_0) - x_0)$. Tem-se:

$$T_1 R_1(x_0, T_1 R_1(x_0), EJ) = J_2(x_0, y_0, J) = \varphi .$$

Nestas decomposições, todas as rotações componentes têm a mesma matriz, a qual coincide com a matriz do deslocamento. Por esta razão, a matriz dum deslocamento recebe o nome de matriz rotação do deslocamento. O facto que apontamos significa que, em todas aquelas decomposições, os eixos das rotações componentes, têm todos a mesma direcção.

10 — Movimento de um sólido invariável

Dá-se o nome de movimento de um sólido invariável S, a uma aplicação μ de I em C, onde I é um intervalo fechado de R^1 e C é a família das representações geométricas de S. Esta aplicação deve obedecer a uma condição que adiante se refere.

Se $t \in I$, t diz-se um instante. I recebe o nome de intervalo de tempo do movimento μ. Se $\mu(t) = \psi_t$, a $\psi_t(S)$ dá-se o nome de configuração de S no instante t, no movimento μ.

Seja μ um movimento do sólido S, de intervalo I. Tome-se em I um instante t_0, e seja $\psi_0(S)$ a configuração de S no instante t_0. Fica então estabelecida uma aplicação de I em F (F família dos deslocamentos do sólido), tal que, $\forall t \in I$, φ_t é o deslocamento do sólido da configuração $\psi_0(S)$ para a configuração $\psi_t(S)$. Desta forma, o movimento μ pode ser representado por:

$$\forall t \in I, \qquad \varphi_t : (x_0, \varphi_t(x_0), J_t),$$

sendo

$$\varphi_t(x) = \varphi_t(x_0) + J_t(x - x_0).$$

A condição atrás referida é que as funções $\varphi_t(x_0)$ e J_t sejam de classe C^2 em I. Estas funções verificam evidentemente as condições $\varphi_{t_0}(x_0) = x_0$, $J_{t_0} = E$.

Como $\det J_t$ é também de classe C^2 em I, e pode apenas tomar os valores ± 1, segue-se que $\det J_t$ é constante. Como $\det J_{t_0} = +1$, vem $\det J_t = +1$ para todo o $t \in I$. É esta circunstância que obriga a que os deslocamentos de um sólido invariável sejam isometrias positivas, já que, dado um deslocamento φ de um sólido, se impõe sempre a condição que seja fisicamente realizável, o que significa que existe um movimento μ, de intervalo I, tal que, existe $t_1 \in I$ para o qual $\varphi_{t_1} = \varphi$.

Diz-se o nome de velocidade, no instante t, do ponto de sólido de posição x_0 em t_0, à derivada em ordem ao tempo de função $\varphi_t(x)$. É então:

$$(14) \qquad \mathbf{v}_t(x) = \mathbf{v}_t(x_0) + \frac{d J_t}{d t}(x - x_0).$$

Obtém-se assim um campo vectorial — campo das velocidades do sólido no instante t — de domínio $\psi_0(S)$. Interessa muitas vezes apresentar o campo das velocidades do sólido S, no instante t, por um campo vectorial de domínio $\psi_t(S)$. Como

$$\varphi_t(x) = \varphi_t(x_0) + J_t(x - x_0),$$

tem-se:

$$\mathbf{v}_t(x) = \mathbf{v}_t(x_0) + \frac{d J_t}{d t} J_t^{-1}[\varphi_t(x) - \varphi_t(x_0)]$$

ou

$$(15) \qquad \mathbf{v}_t(x) = \mathbf{v}_t(x_0) + W_t[\varphi_t(x) - \varphi_t(x_0)],$$

onde $W_t = \dfrac{d J_t}{d t} J_t^{-1}$, é designado por matriz instantânea de rotação.

Supondo o referencial de R^3 ortonormado, é $J_t^T = J_t^{-1}$, donde $W_t = \dfrac{d J_T}{d t} J_t^T$, e portanto $W_t^T = J_t \dfrac{d J_t^T}{d t} = J_t \dfrac{d J_t^{-1}}{d t} = - W_t$. A matriz W_t é anti-simétrica.

a) *Translações.* O movimento μ, de intervalo de tempo I, diz-se um movimento de translação, se para $\forall t \in I$, φ_t é uma translação. Tem-se, então, $J_t = E$, e, de (15), ou (14):

$$\bar{\mathbf{v}}_t(x) = \bar{\mathbf{v}}_t(x_0).$$

Numa translação, todos os pontos têm, num mesmo instante, velocidades iguais. Esta propriedade caracteriza os movimentos de translação, porquanto, se:

$$(16) \qquad \forall t \in I, \quad \forall x \in R^3, \quad \mathbf{v}_t(x) = \mathbf{v}_t(x_0),$$

implica $\dfrac{d J_t}{d t} = 0$, ou $J_t =$ matriz constante. Esta será igual a E porquanto $J_{t_0} = E$.

b) *Rotações em torno de um ponto.* O movimento μ diz-se um movimento de rotação em torno de x_0, se, para $\forall t \in I$, φ_t é uma rotação em torno de x_0.

De (15), tem-se, neste caso:

$$\mathbf{v}_t(x) = W_t[\varphi_t(x) - x_0].$$

Os pontos x que, no instante t, têm velocidade nula, são todos e só aqueles para os quais é:

$$W_t[\varphi_t(x) - x_0] = \mathbf{0},$$

ou seja, são aqueles para os quais $\varphi_t(x) - x_0$ é vector próprio da matriz W_t correspondente ao valor próprio zero. Existe uma e uma só direcção de vectores reais nessas condições — a menos que $W_t = 0$ —, de modo que, em cada instante, existe uma recta $Y_t(x_0, \mathbf{e}_t)$ — onde $W_t \mathbf{e}_t = \mathbf{0}$ — tal que todos os pontos do sólido situados sobre essa recta têm velocidade nula nesse instante. A essa recta dá-se o nome de eixo instantâneo de rotação.

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