Mix *-quantales and the continuous weak order

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Abstract:

The set of permutations on a finite set can be given the lattice structure known as the weak Bruhat order. This lattice structure is generalized to the set of words on a fixed alphabet $\Sigma = \{ x, y, z, \ldots \}$, where each letter has a fixed number of occurrences. These lattices are known as multinomial lattices and, when $\text{card}(\Sigma) = 2$, as lattices of lattice paths. By interpreting the letters $x, y, z, \ldots$ as axes, these words can be interpreted as discrete increasing paths on a grid of a $d$-dimensional cube, with $d = \text{card}(\Sigma)$.

In this talk I’ll explain how to extend this order to images of continuous monotone functions from the unit interval to a $d$-dimensional cube. The lattice so obtained is denoted $L(I^d)$. The key tool used to realize this construction is the quantale $Q\vee(I)$ of join-continuous functions from the unit interval to itself; the construction relies on a few algebraic properties of this quantale: it is involutive (that is, cyclic, non-commutative and *-autonomous, often called a Girard quantale since it is a model of classical linear logic) and it satisfies the mix rule.

We begin developing a structural theory of the lattices $L(I^d)$: they are self-dual, they are generated under infinite joins from their join-irreducible elements, they have no completely irreducible elements nor compact elements.

The colimit of all the $d$-dimensional multinomial lattices embeds into $L(I^d)$ by taking rational coordinates. When $d = 2$, $L(I^d) = Q\vee(I)$ is the Dedekind-MacNeille completion of this colimit. When $d \geq 3$, every element of $L(I^d)$ is a join of meets of elements from this colimit.