Strong affine representations of the polycyclic monoids

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Representations of polycyclic monoids

Definition (Nivat and Perrot, 1970)

The polycyclic monoid $\mathcal{P}_n$ is the monoid with zero defined by the presentation

$$\mathcal{P}_n = \left\langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \mid a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0 \text{ for all } i \neq j \right\rangle.$$

A representation of $\mathcal{P}_n$ is a homomorphism to a symmetric inverse monoid $I(X)$ (the monoid of partial bijections on some set X):

$$\varphi \colon \mathcal{P}_n \to I(X)\,.$$

It suffices to know the images of the generators: let $f_i = \varphi\left(a_i\right)$.

The defining relations of $\mathcal{P}_n$ mean that each f_i is a bijection of the form $f_i \colon X \to X_i \subseteq X$, and the images X_i are pairwise disjoint.

If $X = X_1 \cup \cdots \cup X_n$, then φ is called a strong representation.

Representations of Cuntz algebras

Definition (Cuntz, 1977)

The Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by n pairwise orthogonal isometries on a Hilbert space:

$$\mathcal{O}_n = C^*\left(S_1, \ldots, S_n\right), \text{ where } S_i \in \mathcal{B}\left(\mathcal{H}\right), \ S_i^* S_i = I \text{ and } S_1 S_1^* + \cdots + S_n S_n^* = I.$$

Definition (Bratteli and Jorgensen, 1999)

Permutative representations of $\mathcal{O}_n$: each S_i permutes the elements of an orthonormal basis $\{e_k \colon k \in \mathbb{Z}\}$

$$\forall i \in \{1, \ldots, n\} \ \forall k \in \mathbb{Z} \ \exists \ell \in \mathbb{Z} \colon \ S_i e_k = e_\ell.$$

The index ℓ depends on i and k: let $\ell = f_i\left(e_k\right)$. The definition of $\mathcal{O}_n$ implies that

- each $f_i \colon \mathbb{Z} \to \mathbb{Z}$ is an injective function,
- the sets $f_i\left(\mathbb{Z}\right)$ are pairwise disjoint, and
- $f_1\left(\mathbb{Z}\right) \cup \cdots \cup f_n\left(\mathbb{Z}\right) = \mathbb{Z}$.

Branching function systems

Definition (Bratteli and Jorgensen, 1999)

A branching function system is a tuple $(X; f_1, \ldots, f_n)$, where

- X is an infinite set,
- each $f_i \colon X \to X$ is an injective function,
- the sets $X_i := f_i(X)$ are pairwise disjoint, and
- $X = X_1 \cup \cdots \cup X_n$.

Let us draw an arrow of color i from a to b if $f_i(a) = b$. This way we obtain an edge-colored graph with vertex set X such that

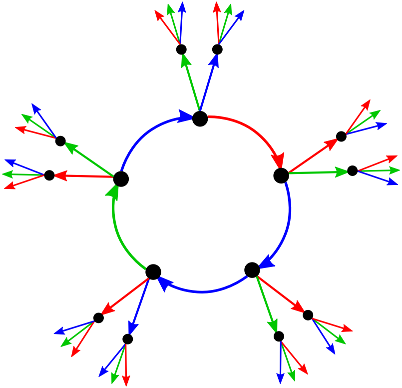
- each vertex has exactly one incoming edge, and
- each vertex has exactly n outgoing edges, one of each of the n colors.

Fact (Lawson, 2009)

If a connected component contains a cycle, then the structure of this component is determined by the order of colors appearing along this cycle.

Branching function systems

A connected component can be described by a word over the set of colors. If two words are cyclic shifts of each other, then they determine the same graph; it is customary to choose the lexicographically smallest one (Lyndon word).



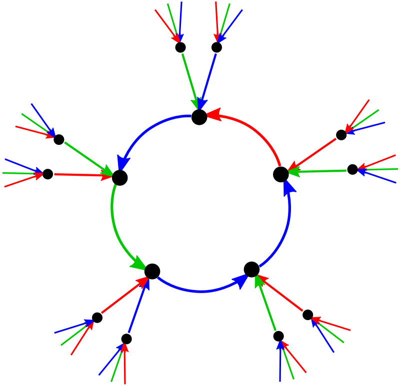
rbbgb ∼ bbgbr ∼ bgbrb ∼ gbrbb ∼ brbbg

Reverse all the arrows

Reversing the arrows, we get the graph of the transformation $R := f_1^{-1} \cup \cdots \cup f_n^{-1}$:

$$R\colon X \to X, \quad R(x) = f_i^{-1}(x) \text{ if } x \in X_i.$$

The vertices on the cycles are the periodic points of the dynamical system $(X; R)$.



A very special case

One-dimensional affine representations: $(\mathbb{Z}; f_1, \ldots, f_n)$, where

- $D = \{d_1, \ldots, d_n\}$ is a complete system of residues modulo n, and
- $f_i\colon \mathbb{Z} \to n\mathbb{Z} + d_i,\ x \mapsto nx + d_i$.
- The edges of the graph are colored/labeled by $d_1, \ldots, d_n$.

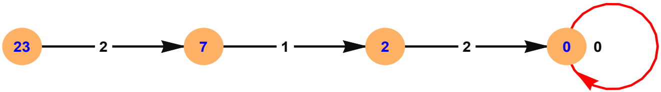
As before, let $R = f_1^{-1} \cup \cdots \cup f_n^{-1}$:

$$R\colon X \to X, \quad R(x) = \frac{x - d_i}{n},$$

where d_i is the unique element of the set D such that $x \equiv d_i \pmod{n}$.

Example

Let $n = 3$ and $D = \{0, 1, 2\}$. The orbit of 23 looks like this:



This orbit provides the ternary representation $23 = \overline{\cdots 000212}_3$.

Strange number systems

Let us write down the labels of the arrows along the orbit of a fixed integer $x \in \mathbb{Z}$:

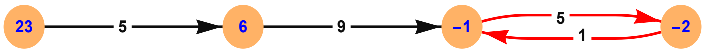
$$x \xrightarrow{a_0} R(x) \xrightarrow{a_1} R(R(x)) \xrightarrow{a_2} R(R(R(x))) \xrightarrow{a_3} \cdots \qquad (a_i \in D)\,.$$

The sequence $a_0, a_1, a_2, a_3, \ldots$ can be interpreted as the "digits" of an n-ary representation of x:

$$x \overset{?!}{=} a_0 + a_1 \cdot n + a_2 \cdot n^2 + a_3 \cdot n^3 + \ldots.$$

Example

Let $n = 3$ and $D = \{1, 5, 9\}$. The orbit of 23 looks like this:



$$\overline{\cdots 15151595}_3 = 5 + 9 \cdot 3 + 5 \cdot 3^2 + 1 \cdot 3^3 + 5 \cdot 3^4 + 1 \cdot 3^5 + 5 \cdot 3^6 + 1 \cdot 3^7 + \cdots$$
$$= 32 + 72 \cdot \left(1 + 3^2 + 3^4 + 3^6 + \ldots\right) = 32 + 72 \cdot \frac{1}{1-9} = 23$$

This always works, if the sequence of digits is periodic!

All orbits are periodic

Let $B_\infty(D)$ denote the set of periodic points of the dynamical system $(\mathbb{Z}; R)$:

$$B_\infty(D) := \left\{x \in \mathbb{Z} : R^\ell(x) = x \text{ for some } \ell \in \mathbb{N}\right\}.$$

Problem

What is the size of B_∞?

Let $\mathcal{I}(D)$ denote the closed interval

$$\left[-\frac{\max D}{n-1}, -\frac{\min D}{n-1}\right].$$

Fact

- $x < \min\mathcal{I} \implies x < R(x) < \max\mathcal{I}$
- $\min\mathcal{I} \le x \le \max\mathcal{I} \implies \min\mathcal{I} \le R(x) \le \max\mathcal{I}$
- $\max\mathcal{I} < x \implies \min\mathcal{I} < R(x) < x$

Corollary

Every orbit is eventually periodic, and $B_\infty(D) \subseteq \mathcal{I}(D) \cap \mathbb{Z}$.

Some motivating results

Fact

The representations corresponding to D and $D+n-1$ are equivalent. Therefore, we can always assume that $0 \leq \min D < n-1$.

Theorem (Bratteli and Jorgensen, 1999; Jones and Lawson, 2012)

Let p be an odd natural number and $D = \{0, p\}$. Then we have

- $B_\infty(D) = \{-p, \ldots, -1, 0\} = \mathcal{I}(D) \cap \mathbb{Z}$;
- *the period of $x \in B_\infty(D)$ equals the order of 2 modulo* $\dfrac{p}{\gcd(x, p)}$;
- *the Lyndon word describing the cycle containing $x \in B_\infty(D)$ is closely related to the digits in the binary expansion of* $\dfrac{x}{p}$.

Theorem (Bratteli and Jorgensen, 1999)

- $B_\infty(0, 1, \ldots, n-1) = \{-1, 0\} = \mathcal{I}(0, 1, \ldots, n-1) \cap \mathbb{Z}$.
- $B_\infty(1, 3, 5) = \{-2, -1\} = \mathcal{I}(1, 3, 5) \cap \mathbb{Z}$.

Arithmetic sequences

Theorem

Let D be an arithmetic sequence $d_1,\ d_1+h,\ d_1+2h,\dots,\ d_1+(n-1)\,h$, where h is a natural number relatively prime to n. Then we have

- $B_\infty(D)=\mathcal{I}(D)\cap\mathbb{Z}$;
- *the Lyndon word describing the cycle containing $x\in B_\infty(D)$ is closely related to the digits in the n-nary expansion of* $\dfrac{x}{h}+\dfrac{d_1}{h(n-1)}$;
- *the period of $x\in B_\infty(D)$ equals the order of n modulo*
$$\frac{h(n-1)}{\gcd\big(x(n-1)+d_1,h(n-1)\big)}.$$

Theorem

For an arbitrary complete system of residues D modulo n, the following two conditions are equivalent:

(i) $B_\infty(D)=\mathcal{I}(D)\cap\mathbb{Z}$;

(ii) $\left\lfloor\dfrac{d_1}{n(n-1)}+\dfrac{d_{i+1}}{n}\right\rfloor=\left\lfloor\dfrac{d_n}{n(n-1)}+\dfrac{d_i}{n}\right\rfloor$ *for* $i=1,\dots,n-1$.

A single periodic point

We start with the simplest arithmetic sequence: $B_\infty(1, \ldots, n) = \{-1\}$.
Now let us modify this by adding n^k to one of the elements.

Theorem

If $D = \left\{1, 2, \ldots, r + n^k, \ldots, n\right\}$, then the number of periodic points is

$$|B_\infty(D)| = \begin{cases} 1, & \text{if } r \notin \{n-2, n-1\}; \\ 2^k, & \text{if } r \in \{n-2, n-1\}. \end{cases}$$

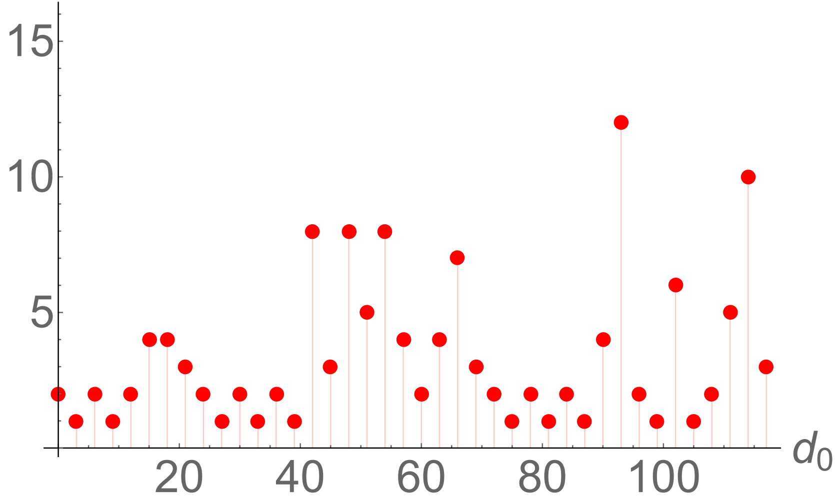
Theorem

If $D = \left\{0, \ldots, n-2, n^k - 1\right\}$, then

$$|B_\infty(D)| = 2^k \quad \text{and} \quad |B_\infty(D+1)| = 1.$$

Experimental results

The number of periodic points for $n = 3$, $D = \{d_0, 1, 2\}$:



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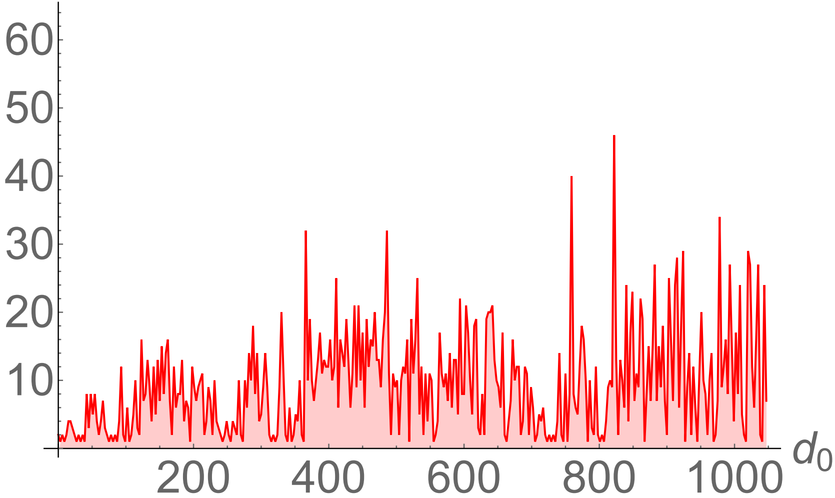

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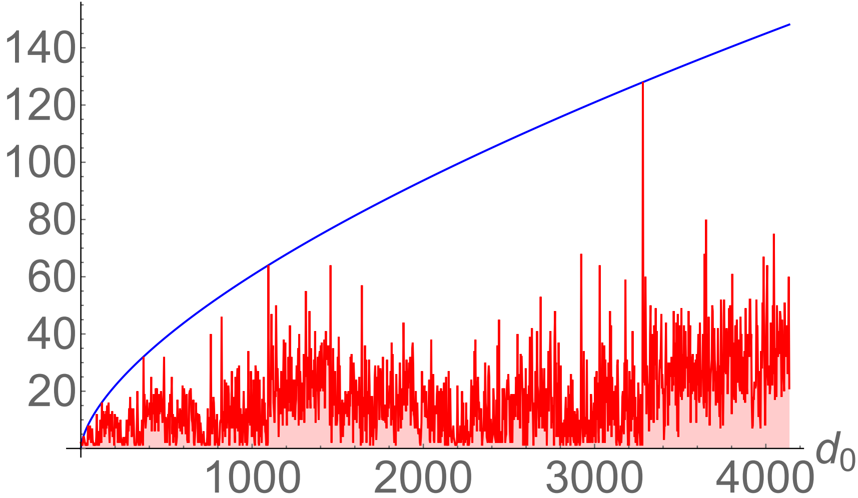

Asymptotics

Problem

What is the asymptotic behaviour of $|B_\infty(d_1, \ldots, d_n)|$ *when one/some/all of the digits* d_i *go to infinity?*

Theorem

If $d_1, \ldots, d_{n-1}$ *are fixed and* $d_n \to \infty$ *(in such a way that* $d_1, \ldots, d_n$ *is a complete system of residues modulo* n*), then*

$$|B_\infty(d_1, \ldots, d_n)| = O\left(d_n^{\log_n 2}\right).$$

Theorem

Let $d_1, \ldots, d_n$ *be an arbitrary complete system of residues modulo* n*, and let* $s \to \infty$ *through integers relatively prime to* n*. Then* $|B_\infty(s \cdot D)|$ *grows linearly with* s*:*

$$\lim_{s\to\infty} \frac{|B_\infty(s \cdot d_1, \ldots, s \cdot d_n)|}{s} = \gcd\left\{d_i - d_j \colon 1 \leq i < j \leq n\right\}.$$

The self-similar tile associated with D

Theorem (Bratteli and Jorgensen, 1999)

If $D = \{d_1, \ldots, d_n\}$ is an arbitrary complete system of residues modulo n, then $B_\infty(D) = -\mathbb{T}(D) \cap \mathbb{Z}$, where

$$\mathbb{T}(D) = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{n^i} : a_i \in D \right\}.$$

Note that $\mathbb{T}(D)$ is a self-similar set (a union of smaller copies of itself):

$$\mathbb{T}(D) = \frac{d_1}{n} + \frac{1}{n} \cdot \mathbb{T}(D) \ \cup \ \cdots \ \cup \ \frac{d_n}{n} + \frac{1}{n} \cdot \mathbb{T}(D).$$

Theorem (Bandt, 1991; Gröchenig and Haas, 1994; Keesling, 1999)

If $D = \{d_1, \ldots, d_n\}$ is an arbitrary complete system of residues modulo n, then

- *$\mathbb{T}(D)$ is a compact set with nonempty interior;*
- *$\mu(\mathbb{T}(D)) = \gcd\{d_i - d_j : 1 \leq i < j \leq n\}$;*
- *the boundary of $\mathbb{T}(D)$ has Lebesgue measure zero.*

Asymptotics

Theorem

$$\lim_{s\to\infty}\frac{\left|B_\infty\left(s\cdot D\right)\right|}{s}=\mu\left(\mathbb{T}\left(D\right)\right)=\gcd\left\{d_i-d_j\colon 1\le i<j\le n\right\}.$$

Proof.

Recall that $B_\infty\left(s\cdot D\right)=-\mathbb{T}\left(s\cdot D\right)\cap\mathbb{Z}$, hence

$$\left|B_\infty\left(s\cdot D\right)\right|=\left|\mathbb{T}\left(s\cdot D\right)\cap\mathbb{Z}\right|=\left|s\cdot\mathbb{T}\left(D\right)\cap\mathbb{Z}\right|=\left|\mathbb{T}\left(D\right)\cap\frac{1}{s}\cdot\mathbb{Z}\right|,$$

which is just the number of rationals of the form $\frac{k}{s}$ $\left(k\in\mathbb{Z}\right)$ in the set $\mathbb{T}\left(D\right)$.

Since this set is Jordan measurable, we have

$$\lim_{s\to\infty}\frac{1}{s}\cdot\left|\mathbb{T}\left(D\right)\cap\frac{1}{s}\cdot\mathbb{Z}\right|=\mu\left(\mathbb{T}\left(D\right)\right).$$

□