Embedding in factorisable restriction monoids

Mária B. Szendrei

Bolyai Institute
University of Szeged

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Restriction semigroups

"restriction" (formerly "weakly *E*-ample") semigroup:
non-regular generalisation of inverse semigroup

$\mathcal{I}_X$ — semigroup of all partial 1-1 transformations on X
$^{-1}$ — unary operation

induced unary operations:

$$\alpha^+ \stackrel{\text{def}}{=} 1_{\text{dom}\,\alpha} \quad \text{and} \quad \alpha^* \stackrel{\text{def}}{=} 1_{\text{im}\,\alpha}$$

every idempotent is of these forms

inverse semigroup $\sim$ Wagner–Preston representation theorem

Restriction semigroups

$\mathcal{PT}_X$ — semigroup of all partial transformations on X

$^+$ — unary operation: $\alpha^+ \stackrel{\text{def}}{=} 1_{\operatorname{dom}\alpha}$

not each idempotent is of this form

Definition

$S = (S; \cdot, ^+)$ is a left restriction semigroup

$\stackrel{\text{def}}{\iff}$ S is isomorphic to a unary subsemigroup of $\mathcal{PT}_X = (\mathcal{PT}_X; \cdot, ^+)$

$\stackrel{\text{def}}{\iff}$ S is a unary semigroup satisfying the identities

$$x^+x = x, \qquad x^+y^+ = y^+x^+,$$
$$(x^+y)^+ = x^+y^+, \quad xy^+ = (xy)^+x$$

Restriction semigroups

Dual of a left restriction semigroup:

$S = (S; \cdot, ^*)$ — right restriction semigroup

Note: $(\mathcal{PT}_X; \cdot, ^*)$ where * is defined by $\alpha^* \stackrel{\text{def}}{=} 1_{\operatorname{im}\alpha}$
is not a right restriction semigroup

Definition

$S = (S; \cdot, ^+, ^*)$ is a restriction semigroup
$\stackrel{\text{def}}{\Longleftrightarrow}$ $(S; \cdot, ^+)$ is left restriction,
$(S; \cdot, ^*)$ is right restriction, and
$E = \{a^+ : a \in S\} = \{a^* : a \in S\}$

last property $\Longleftrightarrow$ S satisfies the identities

$$(x^+)^* = x^+ \quad \text{and} \quad (x^*)^+ = x^*$$

Restriction semigroups

Fact

E forms a semilattice where $e^+ = e^* = e$ $(e \in E)$

E — semilattice of projections of S

$\leq$ — natural partial order on S:

$$a \leq b \overset{\text{def}}{\iff} a = eb \text{ for some } e \in E \ (a, b \in S)$$

$$a \leq b \overset{\text{def}}{\iff} a = be \text{ for some } e \in E \ (a, b \in S)$$

compatible with all three operations

σ — least congruence on S where E is within a class

$=$ least equivalence containing $\leq$

Restriction semigroups

Examples

Reduct of an inverse semigroup S:

$S = (S; \cdot, ^+, ^*)$ where $a^+ \stackrel{\text{def}}{=} aa^{-1}, \; a^* \stackrel{\text{def}}{=} a^{-1}a \; (a \in S)$

Semilattice Y (as a restriction semigroup):

$Y = (Y; \cdot, ^+, ^*))$ where $a^+, a^* \stackrel{\text{def}}{=} a \; (a \in M)$

Monoid T (as a restriction semigroup):

$T = (T; \cdot, ^+, ^*)$ where $t^+, t^* \stackrel{\text{def}}{=} 1 \; (t \in T)$

Y — “semilattice”

T — “monoid”, or “reduced restriction monoid”

Fact

S/σ is reduced

Free restriction semigroups

$F\mathcal{G}(\Omega)$ — free group on Ω
$\mathcal{X}$ — finite connected subgraphs of the Cayley graph of $F\mathcal{G}(\Omega)$
$\mathcal{Y}$ — finite connected subgraphs containing the vertex 1

Fact

$F\mathcal{I}(\Omega) \stackrel{\text{def}}{=} P(F\mathcal{G}(\Omega), \mathcal{X}, \mathcal{Y})$ *is a free inverse semigroup on* Ω

Fountain, Gomes, Gould (2009)

Result

The restriction subsemigroup

$$F\mathcal{R}(\Omega) \stackrel{\text{def}}{=} \{(A, u) \in F\mathcal{I}(\Omega) : u \in \Omega^*\}$$

of $F\mathcal{I}(\Omega)$ *is a free restriction semigroup on* Ω.

Free restriction semigroups

the Cayley graph of $F\mathcal{G}(\Omega)$ is a tree $\Longrightarrow$ $\mathcal{X}$ is a semilattice:

$X \wedge Y$ — least (finite) connected subgraph containing X and Y ($X, Y \in \mathcal{X}$)

Facts

$F\mathcal{I}(\Omega)$ is an inverse subsemigroup in $\mathcal{X} \rtimes F\mathcal{G}(\Omega)$
$F\mathcal{R}(\Omega)$ is a restriction subsemigroup in $\mathcal{X} \rtimes F\mathcal{G}(\Omega)$

Notice:

Ω^* acts on $\mathcal{X}$ by automorphisms, and
$\{(X, u) \in \mathcal{X} \rtimes F\mathcal{G}(\Omega) : u \in \Omega^*\}$
— is a "semidirect product" of $\mathcal{X}$ by Ω^*, and
— is a restriction subsemigroup of $\mathcal{X} \rtimes F\mathcal{G}(\Omega)$ contaning $F\mathcal{R}(\Omega)$

Semidirect product of semilattice by a monoid

Y — semilattice
T — monoid
T acts on Y on the left by automorphisms:

$$a \mapsto {}^{t}a \; (a \in Y,\, t \in T)$$

Definition

$Y \rtimes T \stackrel{\text{def}}{=} Y \times T$ with operations

$$(a, t)(b, u) \stackrel{\text{def}}{=} (a \wedge {}^{t}b, tu)$$

$$(a, t)^{+} \stackrel{\text{def}}{=} (a, 1) \quad \text{and} \quad ({}^{t}a, t)^{*} \stackrel{\text{def}}{=} (a, 1)$$

Note: if T is a subsemigroup in a group G and so we can use t^{-1} within G then the rule for * is also of the usual form:

$$(a, t)^{*} \stackrel{\text{def}}{=} ({}^{t^{-1}}a, 1)$$

Semidirect product of semilattice by a monoid

Facts

1. $E(Y \rtimes T) = Y \times \{1\} \cong Y$
2. *σ of $Y \rtimes T$ is the congruence induced by the second projection, and so $(Y \rtimes T)/\sigma \cong T$*
3. *$Y \rtimes T$ is a monoid iff Y is (i.e., $Y = Y^1$)*
4. *$Y \rtimes T$ gives rise to $Y^1 \rtimes T$ s.t. $Y \rtimes T \leq Y^1 \rtimes T$*

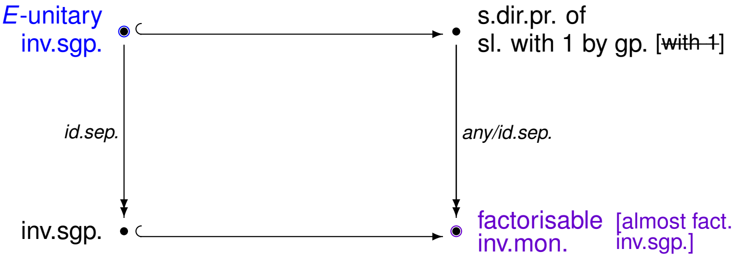
semidirect product of a semilattice by a group

— fundamental role in the structure theory of inverse semigroups

Inverse semigroups and semidirect products

McAlister (1974)
O'Carroll (1976)

Chen, Hsieh (1974)
McAlister, Reilly (1977)
[Lawson (1994)]



Aim:
to generalise some of these results for restriction semigroups

Proper restriction monoids

E-unitary inverse semigroup $\rightsquigarrow$ proper restriction semigroup

S — restriction semigroup

Definition

S is proper

$$\stackrel{\text{def}}{\Longleftrightarrow} \quad \begin{array}{l} a^+ = b^+ \text{ and } a\,\sigma\,b \text{ imply } a = b, \text{ and} \\ a^* = b^* \text{ and } a\,\sigma\,b \text{ imply } a = b \ (a, b \in S) \end{array}$$

Facts

$Y \rtimes T$ and its restriction subsemigroups are proper
in particular, $F\mathcal{R}(\Omega)$ is proper

Proper restriction semigroups

Fountain, Gomes, Gould (2009)

Result

If $\rho \subseteq \sigma$ then $F\mathcal{R}(\Omega)/\rho$ is proper, and each restriction semigroup has such a proper cover for some Ω and ρ.

$$S \cong F\mathcal{R}(\Omega)/\rho_0 \text{ for some } \Omega \text{ and } \rho_0$$

$$\rho \stackrel{\text{def}}{=} \rho_0 \cap \sigma$$

$$C \stackrel{\text{def}}{=} F\mathcal{R}(\Omega)/\rho$$

Note: $C/\sigma \cong \Omega^*$

Factorisable restriction monoids

factorisable inverse monoid $\rightsquigarrow$ factorisable restriction monoid
$\rightsquigarrow$ one-sided factorisable restriction monoid
$\rightsquigarrow$ "almost" ... restriction sgp.

Gomes, Sz. (2007)

S — restriction monoid
E — semilattice of projections of S
$U \stackrel{\text{def}}{=} \{a \in S : a^+ = a^* = 1\}$ — greatest reduced restriction submonoid in S
$R \stackrel{\text{def}}{=} \{a \in S : a^+ = 1\}$

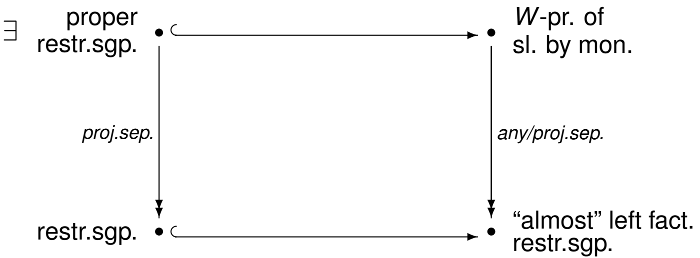
Definition

1. S is factorisable $\stackrel{\text{def}}{\iff}$ $S = EU$ ($\iff$ $S = UE$)
2. S is left factorisable $\stackrel{\text{def}}{\iff}$ $S = ER$

Embedding in “almost” left factorisable restriction semigroups

Fountain, Gomes, Gould (2009) Gomes, Sz. (2007)

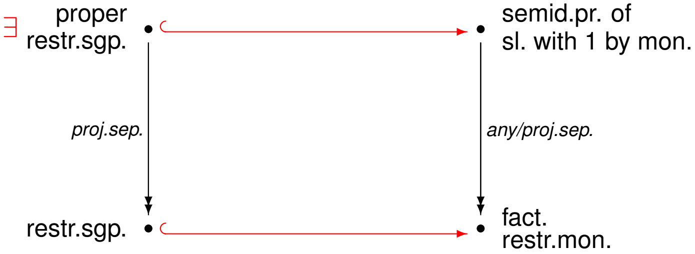
Sz. (2013, 2014)



Main result

Fountain, Gomes, Gould (2009) Gomes, Sz. (2007)

Hartmann, Gould, Sz.



Embedding in factorisable restriction monoids

Theorem

Each restriction semigroup has a proper cover embeddable in a semidirect product of a semilattice by a monoid.

Theorem

Each restriction semigroup is embeddable in a factorisable restriction monoid.

Sketch of the proof:

S — restriction semigroup
$C = F\mathcal{R}(\Omega)/\rho$ — cover of S mentioned above, where

$$S \cong F\mathcal{R}(\Omega)/\rho_0 \text{ and } \rho = \rho_0 \cap \sigma$$

Embedding in factorisable restriction monoids

$F\mathcal{R}(\Omega) \leq \mathcal{X}^1 \rtimes \Omega^*$

extend ρ from $F\mathcal{R}(\Omega)$ to $\mathcal{X}^1 \rtimes \Omega^*$, i.e.,

consider the congruence of $\mathcal{X}^1 \rtimes \Omega^*$ generated by ρ, and prove that its restriction to $F\mathcal{R}(\Omega)$ coincides with ρ

- in the one-sided case, the semilattice component of the W-product was ${}^{\Omega^*}\mathcal{Y}$

a crucial property of the action of Ω^* on $\mathcal{X}$:

for every reduced word $t_1^{\epsilon_1} t_2^{\epsilon_2} \cdots t_n^{\epsilon_n} \in F\mathcal{G}(\Omega)$, where $t_1, t_2, \ldots, t_n \in \Omega^+$ and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ alternate between 1 and -1, the path from 1 to $t_i^{\epsilon_i} t_{i+1}^{\epsilon_{i+1}} \cdots t_n^{\epsilon_n}$ contains the vertex $t_i^{\epsilon_i}$ $(1 \leq i < n)$