On epimorphisms of ordered algebras

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- A pomonoid is a quadruple $(\mathcal{A}, \cdot, 1_A, \leq_A)$, such that $(\mathcal{A}, \cdot, 1_A)$ is a monoid and $(\mathcal{A}, \leq_A)$ is a poset satisfying

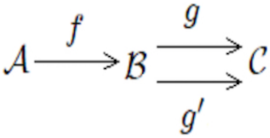
$$\left(a_1 \leq_A a_2 \;\&\; a_1' \leq_A a_2'\right) \Longrightarrow a_1 \cdot a_1' \leq_A a_2 \cdot a_2'$$

- A pomonoid homomorphism is a monotone map

$$f : (\mathcal{A}, \cdot, \leq_A) \longrightarrow (\mathcal{B}, \cdot, \leq_B)$$

  that is also a homomorphism of the underlying monoids.
- Let us denote the category of all pomonoids and their homomorphisms by Pom.

- We call f an epi if it is right cancellative (in Pom), i.e.,
- for every diagram (in Pom)

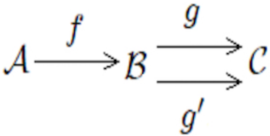


- we have

$$g \circ f = g' \circ f \Longrightarrow g = g'.$$

Motivation

- **Question** Given that f is an epi in Pom, is it necessarily epi in the category Mon of all monoids?
- That is, whether we must have $(g \circ f = g' \circ f \Longrightarrow g = g')$ for every diagram (in Mon):



- The answer is
- YES (Sohail Nasir: Epimorphisms, dominions and amalgamation in pomonoids. Semigroup Forum 90(3), pp 800-809, 2015)

Motivation

- **In one direction**
- An immediate question is the following.
- Do epimorphisms in other varieties (categories) of pomonoids coincide similarly with those of the underlying categories of monoids?
- We don't have an answer.

- **Other direction**
- To what extent the above result can be generalized to ordered algebras vs. the (underlying) unordered algebras?
- It is this latter direction that is the subject of this talk.

Ordered algebras

- **Definition** An ordered Ω-algebra is a triple $(\mathcal{A}, \Omega, \leq_A)$ such that
  - $(\mathcal{A}, \Omega)$ is an Ω-algebra,
  - $(\mathcal{A}, \leq_A)$ is a poset,
  - every $f^A \in \Omega_A$ is monotone, i.e., if f^A has arity n, then
  - $(x_1 \leq_A x_1' \ \& \ x_2 \leq_A x_2' \ \& \ \cdots \& \ x_n \leq_A x_n') \rightarrow f^A(x_1, \ldots, x_n) \leq_A f^A(x_1', \ldots, x_n')$.

Morphisms of ordered algebras

- A homomorphism of ordered algebras is a monotone map, that is also homomorphism of the underlaying algebras.
- A homomorphism $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ is called an order-embedding if $f(x) \leq_B f(x') \Longrightarrow x \leq_A x'$.
- **Fact** Every order-embedding is injective.
- A surjective order-embedding is called an order-isomorphism.
- Epimorphisms are right cancellative homomorphisms, in the sense described earlier.
- **Fact** Every surjective homomorphism is an epi, but the converse is not true.

Conjecture 1

The aim is to prove or disprove the following conjecture.

Conjecture 1 Epimorphisms in a variety of ordered algebras coincide (in the sense described above) with those of the underlying variety of unordered algebras.

We will now find a way to replace the above conjecture by one in a different context.

Dominions

- Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$. Then we define (an ordered subalgebra)

$$\widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C} = \{x \in \mathcal{A} : \forall f, g : \mathcal{A} \longrightarrow \mathcal{B},\ f\,|_{\mathcal{C}} = g\,|_{\mathcal{C}} \Longrightarrow f(x) = g(x)\}$$

- We call $\widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C}$, the (ordered) dominion of $\mathcal{C}$ in $\mathcal{A}$.

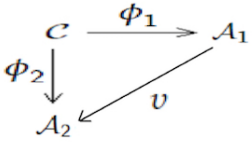
- Treating $\mathcal{C}$ and $\mathcal{A}$ as unordered algebras one gets the analogous definition for $\mathrm{Dom}_{\mathcal{A}}\mathcal{C}$, the unordered dominion of $\mathcal{C}$ in $\mathcal{A}$.

Epimorphisms and dominions, Conjecture 2

- **Fact** $\mathcal{C} \subseteq \mathrm{Dom}_{\mathcal{A}}\mathcal{C} \subseteq \widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C} \subseteq \mathcal{A}$.
- **Fact** $f : (\mathcal{A}, \Omega, \leq_A) \longrightarrow (\mathcal{B}, \Omega, \leq_B)$ is an epi iff $\widehat{\mathrm{Dom}}_{\mathcal{B}}\mathrm{Im}\, f = \mathcal{B}$.
- **Fact** $f : (\mathcal{A}, \Omega) \longrightarrow (\mathcal{B}, \Omega)$ is an epi iff $\mathrm{Dom}_{\mathcal{B}}\mathrm{Im}\, f = \mathcal{B}$.
- So Conjecture 1 will be true if
- **Conjecture 2** $\mathrm{Dom}_{\mathcal{A}}\mathcal{C} = \widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C}$,
  is true.

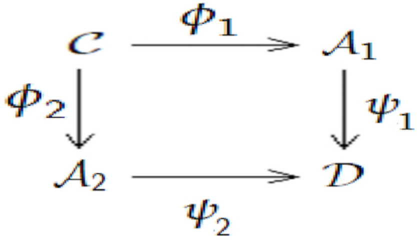
Special amalgams

- We shall next replace Conjecture 2 by yet another one.
- A special amalgam of ordered algebras is a list $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$,
  - where $\mathcal{C}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ are ordered algebras,
  - $\phi_i : \mathcal{C} \longrightarrow \mathcal{A}_i$, $i \in \{1, 2\}$, are order-embeddings, and
  - $\mathcal{A}_1$ is order-isomorphic to $\mathcal{A}_2$, via say $\upsilon : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$, with $\upsilon \circ \phi_1 = \phi_2$.
  - Diagrammatically:



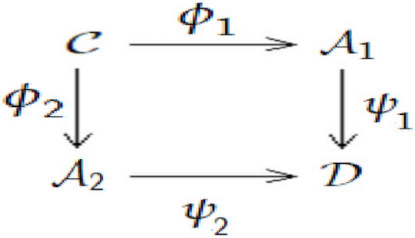
Special amalgams

- Every special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$ is **weakly** embeddable.
- This means the above diagram always completes to a pushout:



Special amalgams

We say that $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \phi_1, \phi_2)$ is (**strongly**) embeddable if the pushout



is also a pullback.

Special amalgams

- Let $\mathcal{C}$ be an ordered subalgebra of an ordered algebra $\mathcal{A}$.
- Take two disjoint order-isomorphic copies $\mathcal{A}_1$ and $\mathcal{A}_2$ via, say,

$$\nu_i : \mathcal{A} \longrightarrow \mathcal{A}_i, i \in \{1, 2\}.$$

- This gives a special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2; \nu_1 |_{\mathcal{C}}, \nu_2 |_{\mathcal{C}})$.
- Indeed every special amalgam can be obtained in this way.

Dominions and special amalgamation

- **Fact** We have

$$\widehat{\mathrm{Dom}}_{\mathcal{A}}\mathcal{C} \cong \widehat{\mathrm{Dom}}_{\mathcal{A}_i} \nu_i\left|_{\mathcal{C}}\right.(\mathcal{C}) = \psi_i^{-1}\left[\psi_1(\mathcal{A}_1) \cap \psi_2(\mathcal{A}_2)\right].$$

- **Fact** The analogue of the above holds in the unordered context.
- **Observation** A special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2)$ of ordered (resp. unordered) algebras is embeddable iff

$$\psi_i^{-1}\left[\psi_1(\mathcal{D}) \cap \psi_2(\mathcal{D})\right] = \nu_i\left|_{\mathcal{C}}\right.(\mathcal{C}),$$

where $\mathcal{D}$ is the respective 'pushout'.
- Hence Conjecture 2 will be true if the following holds.

Conjecture 3

- **Conjecture 3** A special amalgam $(\mathcal{C}; \mathcal{A}_1, \mathcal{A}_2)$ is embeddable in the ordered context iff it is such in the unordered context.
- **Fact** The last conjecture is true for semigroups (monoids) vs. ordered semigroups (monoids).
- **Theorem** Let Ω be a type. Then in the category of all ordered Ω-algebras epis are surjective. (We have a written proof of this.)
- **Theorem** Let Ω be a type. Then in the category of all unordered Ω-algebras epis are surjective.(We have a written proof of this, in fact this is obtained by slightly modifying the above proof.)
- **Corollary** Conjecture 3 is true for any class of all Ω-algebras.

Balanced identities

- An identity is called **balanced** if in both terms, that are used to define it, the number of occurrences of every variable is the same.
- **Theorem** Let $\mathcal{V}$ be a variety of Ω- algebras whose defining identities are balanced. Let $\mathcal{V}'$ be the variety of ordered algebras obtained from $\mathcal{V}$. Then Conjecture 3 is true for $\mathcal{V}$ vs. $\mathcal{V}'$. (We don't have a complete written proof but we think we can write one).
- **Question** What about arbitrary $\mathcal{V}$ and $\mathcal{V}'$ (We don't have any proof, or counter example).

References

This research is being conducted jointly with Professor Boza Tasic.

This talk was motivated by the following articles.

[1] Sohail Nasir: Epimorphisms, dominions and amalgamation in pomonoids. Semigroup Forum DOI: 10.1007/s00233-014-9640-x (2014)

[2] Sohail Nasir: Zigzag theorem for partially ordered monoids. Comm. in Algebra 42, 2559–2583 (2014)

[3] Sohail Nasir: Absolute flatness and amalgamation in pomonoids. Semigroup Forum 82 (3), 504–515 (2011)

THANK YOU