Homological lemmas for Schreier extensions of monoids

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Group actions vs split extensions

An action of a group B on a group X is a group homomorphism $\varphi\colon B \to \mathrm{Aut}(X)$.

Group actions correspond bijectively to split extensions: given a split extension

$$X \xrightarrow{k} A \underset{f}{\overset{s}{\leftrightarrows}} B,$$

the corresponding action is given by

$$b \bullet x = s(b) \cdot x \cdot s(b)^{-1}.$$

Conversely, given an action, the corresponding split extension is obtained via the semidirect product construction.

Monoid actions vs Schreier split epimorphisms

Similarly, an action of a monoid B on a monoid X can be defined as a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$.
To which split extensions do they correspond?

Definition

A split epimorphism $X \xrightarrow{k} A \underset{f}{\overset{s}{\leftrightarrows}} B$ *of monoids is a Schreier split epimorphism if, for any* $a \in A$*, there exists a unique* $x \in Ker(f)$ *such that* $a = x \cdot sf(a)$*.*

Equivalently, there exists a unique map $q\colon A \to X$ such that $a = q(a) \cdot sf(a)$.
Given a Schreier split epimorphism, the corresponding action is defined by

$$b \bullet x = q(s(b) \cdot x).$$

The converse is given, again, by a semidirect product construction.

The Schreier Split Short Five Lemma

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Consider the following commutative diagram, where the two rows are Schreier split extensions:

$$\begin{array}{ccccc} X & \underset{k}{\overset{q}{\leftrightarrows}} & A & \underset{f}{\overset{s}{\leftrightarrows}} & B \\ {\scriptstyle w}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \underset{k'}{\overset{q'}{\leftrightarrows}} & A' & \underset{f'}{\overset{s'}{\leftrightarrows}} & B'. \end{array}$$

We have that

(i) *u is a surjective homomorphism if and only if both v and w are;*

(ii) *u is a monomorphism if and only if both v and w are;*

(iii) *u is an isomorphism if and only if both v and w are.*

Schreier reflexive relations

An internal relation on a monoid B is a submonoid of the product $B \times B$. By considering the homomorphic inclusion

$$R \rightarrowtail B \times B$$

and by composing it with the two projections of the product, we get two parallel homomorphisms

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} B,$$

that are the first and the second projection of the relation.

Definition

An internal reflexive relation of monoids

$$R \underset{p_2}{\overset{p_1}{\underset{\longleftarrow \rho \longrightarrow}{\rightrightarrows}}} B$$

is a Schreier reflexive relation if the split epimorphism (R, B, p_1, ρ) *is a Schreier one.*

Mal'tsev-type properties

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Any Schreier reflexive relation is transitive.
It is a congruence if and only if Ker(p_1) is a group.

Example

The usual order between natural numbers:

$$\mathcal{O}_{\mathbb{N}} \underset{p_2}{\overset{p_1}{\underset{\longleftarrow \rho \longrightarrow}{\longrightarrow}}} \mathbb{N},$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\},$$

is a Schreier order relation.

Special Schreier extensions

Definition

A homomorphism $f\colon A \to B$ is special Schreier if the kernel congruence

$$Eq(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} A \quad \text{with } \langle 1,1\rangle\colon A \to Eq(f)$$

is a Schreier congruence.

This is equivalent to having a partial division on A: if $f(a_1) = f(a_2)$, then there exists a unique $x \in \mathrm{Ker}(f)$ such that $a_2 = x \cdot a_1$.

In particular, $\mathrm{Ker}(f)$ is a group.

If $f\colon A \to B$ is a surjective special Schreier homomorphism, then it is the cokernel of its kernel. Hence we get an extension

$$X \xrightarrow{k} A \xrightarrow{f} B.$$

The special Schreier Short Five Lemma

Theorem (Bourn, Martins-Ferreira, Montoli, S.)

Consider the following commutative diagram, where the two rows are special Schreier extensions:

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \overset{f}{\twoheadrightarrow} & B \\ {\scriptstyle w}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[k']{} & A' & \underset{f'}{\twoheadrightarrow} & B'. \end{array}$$

We have that

(i) *u is a surjective homomorphism if and only if both v and w are;*

(ii) *u is a monomorphism if and only if both v and w are;*

(iii) *u is an isomorphism if and only if both v and w are.*

The special Schreier Nine Lemma

Theorem

Consider the following commutative diagram, where the three columns are special Schreier extensions:

$$
\begin{array}{ccccc}
N & \xrightarrow{\eta} & H & \xrightarrow{\lambda} & K \\
{\scriptstyle l}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\
X & \xrightarrow{\sigma} & Y & \xrightarrow{\varphi} & Z \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C.
\end{array}
$$

1. *If the first two rows are special Schreier extensions, then the lower also is;*
2. *if the last two rows are special Schreier extensions, then the upper also is;*
3. *if $\varphi\sigma = 0$ and the first and the last rows are special Schreier extensions, then the middle also is.*