International Conference of Semigroups and Automata

Celebrating the 60th birthday of **Jorge Almeida** & **Gracinda Gomes**

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Local finiteness for Green relations in (I-)semigroup varieties

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2 ::: Business as usual

SMG: variety of all **semigroups**, ie type (2) algebras satisfying

$$x(yz) = (xy)z\,.$$

CR: variety of all **completely regular semigroups**, ie type $(2,1)$ algebras satisfying

$$x(yz) = (xy)z\,,\ (x')' = x\,,\ xx'x = x\,,\ xx' = x'x\,.$$

INV: variety of all **inverse semigroups**, ie type $(2,1)$ algebras satisfying

$$x(yz) = (xy)z\,,\ (x')' = x\,,\ xx'x = x\,,\ xx'yy' = yy'xx'\,.$$

$\mathcal{I}$**SMG**: variety of all ***I*-semigroups**, ie type $(2,1)$ algebras satisfying

$$x(yz) = (xy)z\,,\ (x')' = x\,,\ xx'x = x\,.$$

3 ::: What about?

For a variety **V**:

V is **locally finite** if all its finitely generated members are finite.

For a variety **V** and $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$:

V is **locally $\mathcal{K}$-finite** if each finitely generated $S \in \mathbf{V}$ has finitely many $\mathcal{K}$-classes.

loc $\mathcal{J}$-finite

loc $\mathcal{D}$-finite

loc $\mathcal{L}$-finite   loc $\mathcal{R}$-finite

loc $\mathcal{H}$-finite

4 ::: Actually...

For $\mathbf{V} \in \mathscr{L}(\mathbf{INV})$:

$$\mathbf{V} \text{ locally } \mathcal{L}\text{-finite} \iff \mathbf{V} \text{ locally } \mathcal{R}\text{-finite} \iff \mathbf{V} \text{ locally } \mathcal{H}\text{-finite}\,.$$

For $\mathbf{V} \in \mathscr{L}(\mathbf{SMG})$ or $\mathbf{V} \in \mathscr{L}(\mathbf{CR})$ or $\mathbf{V} \in \mathscr{L}(\mathbf{INV})$:

$$\mathbf{V} \text{ locally } \mathcal{D}\text{-finite} \iff \mathbf{V} \text{ locally } \mathcal{J}\text{-finite}\,.$$

$\mathscr{L}(\mathbf{SMG})$, $\mathscr{L}(\mathbf{CR})$:

$$\begin{tikzcd}
 & \text{loc } \mathcal{D}\text{-fin} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{loc } \mathcal{L}\text{-fin} \arrow[dr, dash] & & \text{loc } \mathcal{R}\text{-fin} \arrow[dl, dash] \\
 & \text{loc } \mathcal{H}\text{-fin} &
\end{tikzcd}$$

$\mathscr{L}(\mathbf{INV})$:

$$\begin{tikzcd}
\text{loc } \mathcal{D}\text{-fin} \arrow[d, dash] \\
\text{loc } \mathcal{H}\text{-fin}
\end{tikzcd}$$

5 ::: Local $\mathcal{K}$-finiteness and varieties operators

V and **W** varieties of I-semigroups, $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}\}$:

$$\mathbf{V} \text{ and } \mathbf{W} \text{ locally } \mathcal{K}\text{-finite} \Longrightarrow \mathbf{V} \vee \mathbf{W} \text{ locally } \mathcal{K}\text{-finite}\,.$$

but

Every semigroup can be embedded in a simple/bisimple semigroup.

S an I-semigroup, $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}\}$:

$$S \text{ finitely many } \mathcal{K}\text{-classes} \Longrightarrow \langle S \rangle \text{ locally } \mathcal{K}\text{-finite}\,.$$

Not true for $\mathcal{K} = \mathcal{D}$:

B, the bicyclic monoid, is $\mathcal{D}$-finite, whereas $\langle B \rangle = \langle FIS_a \rangle$ is not locally $\mathcal{D}$-finite.

6 ::: Within **CR**

CR is locally $\mathcal{D}$-finite.

The variety **BG** of all completely regular semigroups in which $\mathcal{H}$ is a congruence (cryptogroups) is locally $\mathcal{H}$-finite.

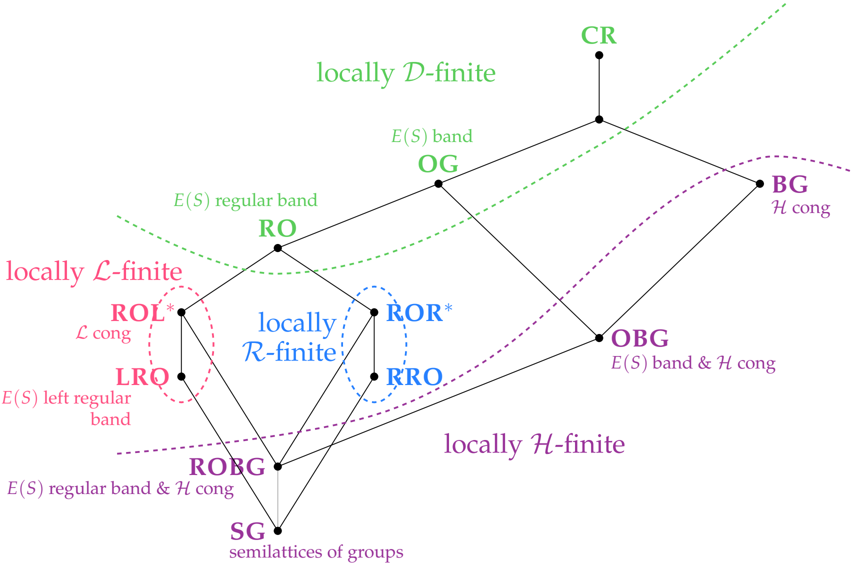
The variety **LRO** of all left regular orthogroups (completely regular & idempotents form a left regular band) is locally $\mathcal{L}$-finite.

The semigroup $P = (\mathbb{Z}, \circ)$ with

$$m \circ n = \begin{cases} m + n & \text{if } m \text{ is even} \\ m & \text{if } m \text{ is odd} \end{cases}$$

belongs to **LRO**, is finitely generated, has finitely many $\mathcal{L}$-classes but infinitely many $\mathcal{R}$-classes.

7 ::: The **CR** picture



8 ::: Within **INV**

$\langle B \rangle = \langle FIS_a \rangle$ is not locally $\mathcal{J}$-finite.

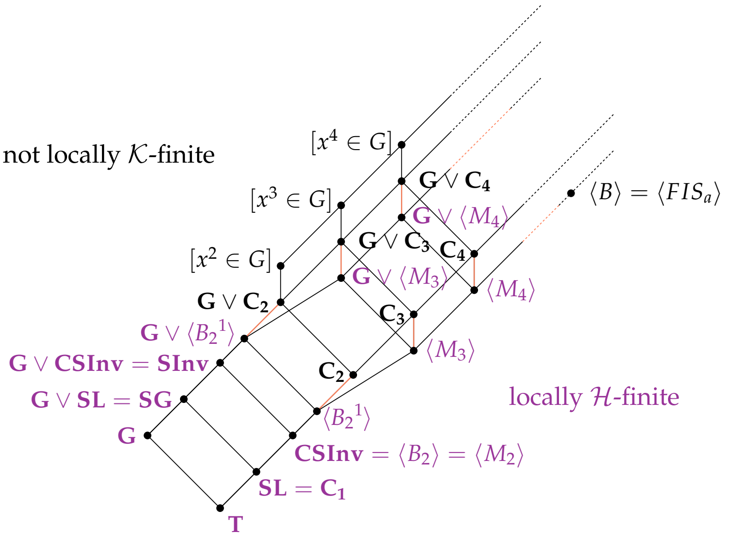
$\left\langle B_2^1 \right\rangle$ and $\langle M_n \rangle$, for any positive integer n, are locally $\mathcal{H}$-finite.

where $\quad B_2^1 = \mathrm{InvM}\left\langle a \mid a^2 = 0 \right\rangle$
$M_n = FIS_a / I_n$, I_n is the ideal of FIS_a generated be a^n.

$\mathbf{C_2} = [x^2 = x^3]$ is not locally $\mathcal{J}$-finite.

$T = \mathrm{InvS}\langle A \mid u^2 = 0 \ (u \in (A \cup A^{-1})^+, \ \overline{u} \neq 1)\rangle$, with $A = \{a, b, c\}$, belongs to $\mathbf{C_2}$, contains the prefixes of Morse and Hedlund's square-free infinite word and no two such elements are $\mathcal{J}$-related.

9 ::: The **INV** picture



in M. Petrich, "Inverse Semigroups", Pure and Applied Mathematics, John Wiley & Sons, 1984.

10 ::: Within **SMG**

$(\mathbb{N}, +) \in \mathbf{V} \implies \mathbf{V}$ not locally $\mathcal{J}$-finite.

$\mathbf{B_{m,n}} = [x^m = x^{m+n}]$: Burnside varieties

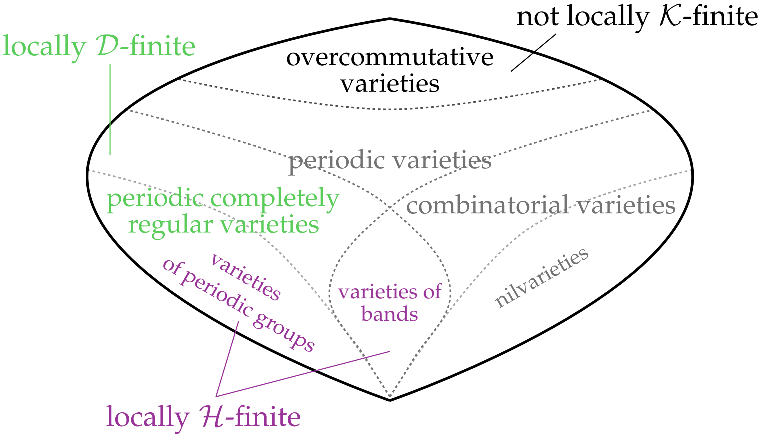
$\mathbf{B_{1,n}} = [x = x^{n+1}]$ with $n = 1, 2, 3, 4, 6$ are locally $\mathcal{H}$-finite;
remaining are at least locally $\mathcal{D}$-finite.

but

$\mathbf{C_2} = [x^2 = x^3] = \mathbf{B_{2,1}}$ is not locally $\mathcal{J}$-finite
and so $\mathbf{B_{m,n}}$, with $m \geq 2$, are not locally $\mathcal{J}$-finite.

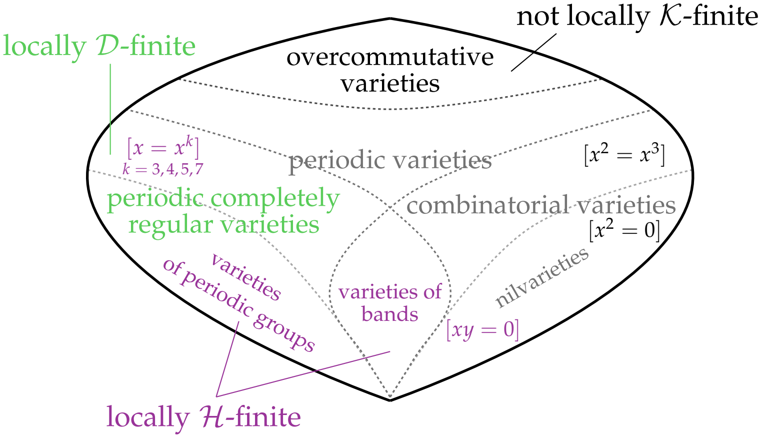
$\mathbf{V} = [u = 0]$ is either $\mathcal{H}$-locally finite or not locally $\mathcal{J}$-finite.

11 ::: The **SMG** picture



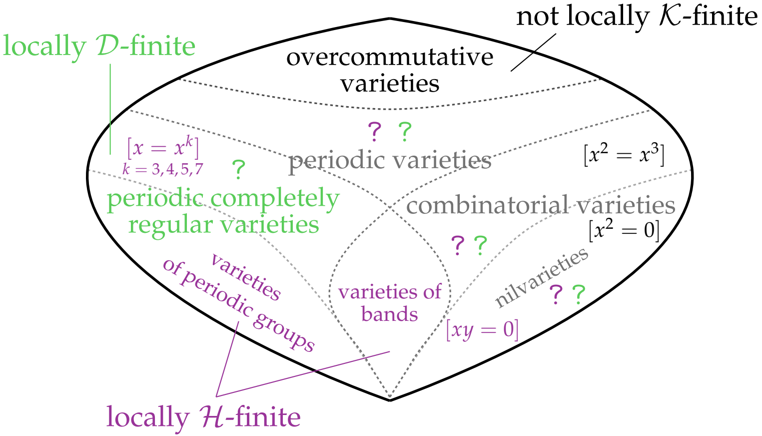
in L. N. Shevrin, B. M. Vernikov, M. V. Volkov, *Lattices of Semigroup Varieties*, Russian Mathematics (Iz. VUZ), **50** No. 3 (2009), 1–28.

12 ::: The **SMG** picture



in L. N. Shevrin, B. M. Vernikov, M. V. Volkov, *Lattices of Semigroup Varieties*, Russian Mathematics (Iz. VUZ), **50** No. 3 (2009), 1–28.

13 ::: The **SMG** picture



in L. N. Shevrin, B. M. Vernikov, M. V. Volkov, *Lattices of Semigroup Varieties*, Russian Mathematics (Iz. VUZ), **50** No. 3 (2009), 1–28.

14 ::: Thank you!

P. Silva, F. Soares,
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arXiv: 1606.03866