Introduction

Regular Rings

Biorder Ideals of Regular Rings

References

Lattice of biorder ideals of regular rings

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Abstract

- Here the left [right] biorder ideals ω^l $[\omega^r]$ of regular rings are defined.
- It is shown that these ideals form a complemented modular lattices Ω_L and Ω_R .
- We also discuss the basis and order of these lattices.

Biordered sets

A partial algebra E is a set together with a partial binary operation on E. The domain of the partial binary operation will be denoted by D_E. On E we define

$$\omega^r = \{(e,f) : fe = e\}\omega^l = \{(e,f) : ef = e\}$$

also., $\mathcal{R} = \omega^r \cap (\omega^r)^{-1}$, $\mathcal{L} = \omega^l \cap (\omega^l)^{-1}$, $and\, \omega = \omega^r \cap \omega^l$

Definition 1

Let E be a partial algebra. Then E is a biordered set if the following axioms and their duals hold:

1. ω^r and ω^l are quasi orders on E and
$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$$
2. $f \in \omega^r(e) \Rightarrow f\mathcal{R}fe\omega e$
3. $g\omega^l f \, and \quad f,g \in \omega^r(e) \Rightarrow ge\omega^l fe.$
4. $g\omega^r f\omega^r e \Rightarrow gf = (ge)f$
5. $g\omega^l f \text{ and } f,g \in \omega^r(e) \Rightarrow (fg)e = (fe)(ge).$

Sandwitch set

Let $\mathcal{M}(e,f)$ denote the quasi ordered set $(\omega^l(e) \cap \omega^r(f), <)$ where $<$ is defined by $g < h \Leftrightarrow eg\omega^r eh, \text{ and } gf\omega^l hf$. Then the set

$$S(e,f) = \{h \in M(e,f) : g < h \text{ forall } g \in M(e,f)\}$$

is called the sandwitch set of e and f.

- $f, g \in \omega^r(e) \Rightarrow S(f,g)e = S(fe, ge)$

The biordered set E is said to be regular if $S(e,f) \neq \emptyset \ \forall e, f \in E$.

- If S is a regular semigroup, then $E(S)$, the set of all idempotenrs of S is a regular biordered set.

Definition 2

For $e \in E$, $\omega^r(e)$ $[\omega^l(e)]$ are principle right [left] ideals and $\omega(e)$ is a principal two sided ideal and these ideals are called biorder ideals generated by e.

Definition 3

Let e and f are idempotents in a semigroup S, then an e-sequence from e to f is a finite sequence $e = e_0, e_1, \cdots, e_n = f$ of idempotents such that $e_{i-1}(\mathcal{L} \cup \mathcal{R})e_i$ for $i = 1, \cdots, n$.

If there exists an E-sequence from e to f, then $d(e, f)$ is the length of the shortest E-sequence from e to f.

Modular Lattice

- A lattice is a partially ordered set in which each pair of elements has a least upper bound and a greatest lower bound.
- A lattice is called modular (or a Dedekind lattice) if the modular law holds in it: $a \leq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$.
- a lattice is bounded if it has both a maximum element and a minimum element. We use the symbols 0 and 1 to denote the minimum element and maximum element of a lattice.
- A bounded lattice L is said to be complemented if for each element a of L, there exists at least one element b such that $a \vee b = 1$ and $a \wedge b = 0$.

Definition 4

Two elements a and b of a lattice L are said to be perspective (in symbols $a \sim b$) if there exists x in L such that $a \vee x = b \vee x, a \wedge x = b \wedge x = 0$ and such an element x is called an axis of perspective.

Definition 5

Let L be a complemented modular lattice with 0 and 1. By a basis of L we mean a system $(a_i : i = 1, \cdots, n)$ of n elements in L such that $a_i : i = 1, \cdots, n$ are independant and $a_1 \cup a_2 \cup \cdots a_n = 1$.

A basis is called homogeneous if its elements are pairwise perspective. The number of elements in a basis is called the order of the basis and a lattice is said to be of order n if it admits a homogenious basis of order n

Regular Rings and Ideals

A ring $(R, +, \cdot)$ is called regular if for every $a \in R$ there exists an element a' such that $aa'a = a$. A subset A of a ring $\mathcal{R}$ is called right ideal in case

$$x + y \in A, \; xz \in A$$

for each $x, y \in A$ and $z \in \mathcal{R}$.
If R is a ring and $\mathbf{a} \subset \mathbf{R}$ is a right ideal then $\mathbf{a}$ has a unique least extension $\langle a \rangle_r$ containing $\mathbf{a}$. Similarly we have the unique left ideal $\langle a \rangle_l$ and two sided ideal $\langle a \rangle$ containing $\mathbf{a}$.

Definition 6

A principal right [left] ideal is one of the from $\langle a \rangle_r [\langle a \rangle_l]$. The class of all principal right [left] ideals will be denoted by $\bar{R}_{\mathcal{R}}$ $[\bar{L}_{\mathcal{R}}]$.

John von Neumann describes the structure of principal ideals of a regular ring, here we recall some of those results.

Lemma 2.1

Let $\mathcal{R}$ be a ring, $e \in \mathcal{R}$, then

- e *is idempotent if and only if* $(1-e)$ *is idempotent.*
- $\langle e \rangle_r$ *is the set of all* x *such that* $x = ex$ *is a principal right ideal.*
- $\langle e \rangle_r$ *and* $\langle 1-e \rangle_r$ *are mutual inverses.*
- *If* $\langle e \rangle_r = \langle f \rangle_r$ *and if* $\langle 1-e \rangle_r = \langle 1-f \rangle_r$ *where* e *and* f *are idempotents, then* $e = f$.

Theorem 1

Two right ideals a and b are inverses if and only if there exists an idempotent e such that $a = \langle e \rangle_r$ and $b = \langle 1 - e \rangle_r$.

Theorem 2

The following statements are equivalent

1. Every principal right ideal $\langle a \rangle_r$ has an inverse right ideal.
2. For every a there exists an idempotent e such that $\langle a \rangle_r = \langle e \rangle_r$.
3. For every a there exists an element x such that $axa = a$.
4. For every a there exists an idempotent f such that $\langle a \rangle_l = \langle f \rangle_l$.
5. Every principal left ideal $\langle a \rangle_l$ has an inverse left ideal.

Theorem 3

The set $\bar{R}_{\mathcal{R}}$ is a complemented, modular lattice partially ordered by $\subset$, the meet being $\cap$ and join $\cup$, its zero is $\langle 0 \rangle_r$ and its unit is $\langle 1 \rangle_r$.

Biorder Ideals of R

In a regular ring R, every principal right ideal is generated by an idempotent. Let $(E_R, \cdot)$ denote the set of all multiplicative idempotents in the ring R. Then $(E_R, \cdot)$ is a regular biordered set with quasiorders ω^r and ω^l.
Note that $\omega^r(e)$ $[\omega^l(e)]$ are right [left] ideals of the ring R and are called the biorder ideals of the ring R.

Proposition 1

Let e and f be idempotents in a regular ring R. Then the following holds.

1. $e\omega^l f$ if and only if $(1-f)\omega^r(1-e)$
2. $e\omega^r f$ if and only if $(1-f)\omega^l(1-e)$

Corollary 1

Let e and f be idempotents in the ring R. Then

1. $\omega^l(e) = \omega^l(f)$ if and only if $\omega^r(1-e) = \omega^r(1-f)$
2. $\omega^r(e) = \omega^r(f)$ if and only if $\omega^l(1-e) = \omega^l(1-f)$

Remark 1

Let R be a regular ring with $ef = 0$ for every $e, f \in E_R$, then it is easy to observe the following:

1. The only idempotent in $M(e, f)$ is $\{0\}$
2. $e\omega(1-f)$

Lemma 3.1

Let R be a regular ring and $e, f \in E_R$ such that $M(e, f) = \{0\}$, then $ef = 0$.

Proof.

Let $M(e, f) = \{0\}$. Since R is regular, the element $ef \in R$ has an inverse $x \in R$ so that

$$(ef)x(ef) = ef$$

$$x(ef)x = x.$$

Let $g = fxe$, then g is an idempotent and $g \in M(e, f)$ so $g = 0$, by hypothesis. Hence

$$ef = (ef)x(ef) = e(fxe)f = (eg)f = 0$$

□

Lemma 3.2

Let $e, f, g \in E_R$ with $ef = fe = 0$. Then $e + f$ is an idempotent and the following holds.

1. $e\omega(e+f)$ *and* $f\omega(e+f)$
2. *If* $e\omega^l g$ *and* $f\omega^l g$*, then* $(e+f)\omega^l g$
3. *If* $e\omega^r g$ *and* $f\omega^r g$*, then* $(e+f)\omega^r g$

Proof.

Given $e, f \in E_R$ with $ef = fe = 0$, then $(e+f)^2 = e^2 + ef + fe + f^2 = e + f$.

- $e(e+f) = e^2 + ef = e + ef = e$, and $(e+f)e = e^2 + fe = e + fe = e$. Thus $e\omega(e+f)$. Similarly, we can prove $f\omega(e+f)$.
- Given $e\omega^l g$ and $f\omega^l g$. Therefore, $(e+f)g = eg + fg = e + f$ i.e., $(e+f)\omega^l g$.

The proof of (3) is similar. □

Lemma 3.3

Let $e, f \in E_R$. *Then* $\omega^r(e) \cup \omega^r(f) = \omega^r(e + f'')$ *where* $f''\mathcal{R}f'$ *and* $f' = (1 - e)f$.

Denote by Ω_R the class of all principal ω^r-ideals and by Ω_L the class of all principal ω^l-ideals. In the light of the above lemma we have the following theorem.

Theorem 4

Ω_R is closed with respect to the operation $\cup$ defined in Ω_R.

Annihilators in ω^r and ω^l-ideals.

Definition 7

For every ω^r-ideal we define

$$(\omega^r(e))^L = \{y\colon yz = 0 \text{ for every } z \in \omega^r(e)\}$$

and for every ω^l-ideal,

$$(\omega^l(e))^R = \left\{y\colon zy = 0 \text{ for every } z \in \omega^l(e)\right\}$$

then $(\omega^r(e))^L$ is a left ideal and $(\omega^l(e))^R$ is a right ideal.

Proposition 2

For $e \in E_R$, $(\omega^l(e))^R$ is a principal ω^r-ideal and $(\omega^r(e))^L$ is a principal ω^l-ideal. In fact, $(\omega^l(e))^R = \omega^r(1-e)$ and $(\omega^r(e))^L = \omega^l(1-e)$.

Proof.

$$
\begin{aligned}
\omega^r(e) &= \{g\colon eg = g\} \\
&= \{g\colon (1-e)g = 0\} \\
&= \{g\colon u(1-e)g = 0;\ \text{for every } u \in E_R\} \\
&= \left\{g\colon\ \text{for every } h \in \omega^l(1-e), hg = 0\right\}
\end{aligned}
$$

where $h = u(1-e)$. Since $h(1-e) = u(1-e)(1-e) = u(1-e) = h$ we have $h \in \omega^l(1-e)$. Thus $\omega^r(e) = (\omega^l(1-e))^R$. □

Lemma 3.4

Let $e, f \in E_R$ *and* $\omega^r(e)$ *and* $\omega^r(f)$ *are ideals generated by* e *and* f*, then*

1. $\omega^r(e) \subset \omega^r(f) \Rightarrow (\omega^r(e))^L \supset (\omega^r(f))^L$
2. $\omega^r(e) = (\omega^r(e))^{LR}$ *and* $(\omega^r(e))^L = (\omega^r(e))^{LRL}$

In the following proposition we establish the relation between Ω_L and Ω_R by using the relation between principal ω-ideals and their annihilators.

Proposition 3

Let R be a regular ring and E_R the set of idempotents in R. Let Ω_L and Ω_R denote the lattice of principal ω^l-ideals and principal ω^r-ideals of E_R. Define ϕ and ψ on Ω_L and Ω_R by

$$\phi(\omega^l(e)) = (\omega^l(e))^R \text{ and } \psi(\omega^r(e)) = (\omega^r(e))^L$$

then ϕ and ψ are mutually inverse anti-isomorphisms.

Lemma 3.5

Let $\omega^r(e)$ and $\omega^r(f)$ be principal right ω-ideals generated by e and f. Then $(\omega^r(e) \cup \omega^r(f))^L = (\omega^r(e))^L \cap (\omega^r(f))^L$.

Lemma 3.6

For two principal ω^r-ideals, $\omega^r(e)$ and $\omega^r(f)$ their intersection is also a principal ω^r-ideal.

For any idempotent $e \in E_R$, $\omega^r(e) \cup \omega^r(1-e) = \omega^r(e+1-e) = \omega^r(1) = E_R$ and $\omega^r(e) \cap \omega^r(1-e) = \{0\}$. Thus $\omega^r(e)$ and $\omega^r(1-e)$ are complements of each other in the lattice of all principal right ω-ideals. Similarly, $\omega^l(e)$ and $\omega^l(1-e)$ are complements of each other in the lattice of all principal left ω-ideals of E_R.
Thus we have the following theorem.

Theorem 5

Let R be a ring then the set of all principal ω^l-ideals Ω_L and the set of all principal ω^r-ideals Ω_R of R are complemented, modular lattices ordered by the relation $\subset$, the meet being $\cap$ and the join $\cup$; its zero is 0, and its unit is $\omega^l(1)[\omega^r(1)]$.

Order of the complemented modular lattices.

Lemma 3.7

Let $\omega^l(e)$ and $\omega^l(f)$ be in Ω_L. Then $\omega^l(e)$ and $\omega^l(f)$ are complements in Ω_L if and only if there exists an idempotent h such that $\omega^l(e) = \omega^l(h)$ and $\omega^l(f) = \omega^l(1-h)$.

Proposition 4

For $e \in E_R$, $(\omega^l(e))^R$ is a principal ω^r-ideal and $(\omega^r(e))^L$ is a principal ω^l-ideal. In fact, $(\omega^l(e))^R = \omega^r(1-e)$ and $(\omega^r(e))^L = \omega^l(1-e)$.

Two elements of a lattice are said to be in perspective if they have a common complement. For idempotents e and f, we define $d_l(e,f)$ to be the length of the shortest E-sequence from e to f, which start with the $\mathcal{L}$ relation and $d_r(e,f)$ to be the length of the shortest E-sequence from e to f which start with the $\mathcal{R}$ relation.

Now we describe perspectivity of two members of Ω_L in a regular ring in terms of the d_l function as follows:

Lemma 3.8

Let $\omega^l(e)$ and $\omega^l(f)$ be biorder ideals in Ω_L. Then $\omega^l(e)$ and $\omega^l(f)$ are perspective in Ω_L if and only if $1 \leq d_l(e, f) \leq 3$.

Definition 8

Let Ω_L be a complemented modular lattice with zero 0 and unit $\omega^l(1)$. A basis of Ω_L is a collection $(\omega^l(e_i), i = 1, 2, \ldots n) \in \Omega_L$ such that $(\omega^l(e_i) \colon i = 1, 2, \ldots, n)$ are independent and $\omega^l(e_1) \cup \ldots \omega^l(e_n) = \omega^l(1)$. The number of elements in a basis is called the order of the basis. Further, a basis is homogeneous if its elements are pairwise perspective.

Theorem 6

Let R be regular ring with $M(e_i, e_j) = \{0\}$ for $i \neq j$ and $d_l(e_i, e_j) \leq 3$. Then the complemented, modular lattice Ω_L is of order n.

References

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References

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