Homogeneous Bands

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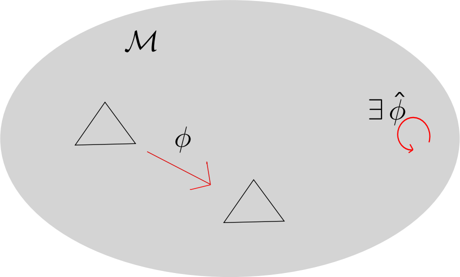
Homogeneity

Definition

A countable (first order) structure $\mathcal{M}$ is *homogeneous* if every isomorphism between finitely generated substructures extends to an automorphism of $\mathcal{M}$.



Some key classifications

- (Droste, Kuske, Truss (1999)) A non-trivial homogeneous (lower) semilattice is isomorphic to either $(\mathbb{Q}, <)$, the universal semilattice or a semilinear order.
- (Schmerl (1979)) Classified homogeneous posets:
  i) $\mathcal{A}_n$, the antichain of n elements;
  ii) $\mathcal{B}_n = \mathcal{A}_n \times \mathbb{Q}$ with partial order,
  $$(a, p) < (b, q) \text{ if and only if } a = b \text{ and } p < q \text{ in } \mathbb{Q},$$
  the union of n incomparable copies of $\mathbb{Q}$;
  iii) $\mathcal{C}_n = \mathcal{A}_n \times \mathbb{Q}$ with partial order
  $$(a, p) < (b, q) \text{ if and only if } p < q \text{ in } \mathbb{Q},$$
  a chain of antichains;
  iv) $\mathbb{P}$, the generic poset,

  where $n \in \mathbb{N}^* = \mathbb{N} \cup \{\aleph_0\}$.

The basics

- An element e is an **idempotent** if $e^2 = e$. A **band** B is a semigroup in which every element is an idempotent. A **semilattice** is a commutative band.

- We may define a partial order $\leq$ on B, known as the **natural order**, by
$$e \leq f \Leftrightarrow ef = fe = e.$$

- If Y is a commutative band then $(Y, <)$ is a lower semilattice, with $a \wedge b = ab$.

Motivating question: Given a homogeneous poset P, does there exist a homogeneous band B such that $(B, <)$ is isomorphic to P?

Rectangular bands

- A **rectangular band** is a band B satisfying

$$efe = e \text{ for all } e, f \in B.$$

- A rectangular band with a single $\mathcal{R}$-class ($\mathcal{L}$-class) is called a **right (left) zero band**.

Proposition

Let I and J be arbitrary sets. Then $B_{I,J} = (I \times J, \cdot)$ forms a rectangular band, with operation given by

$$(i,j) \cdot (k,\ell) = (i,\ell).$$

Moreover every rectangular band is isomorphic to some $B_{I,J}$. The natural order on $B_{I,J}$ is an anti-chain on $|I| \cdot |J|$ elements, and the Greens relations are:

$$(i,j)\,\mathcal{R}\,(k,\ell) \Leftrightarrow i = k \text{ and } (i,j)\,\mathcal{L}\,(k,\ell) \Leftrightarrow j = \ell.$$

Homogeneous rectangular bands

Proposition

$B_{I,J} \cong B_{I',J'}$ if and only if $|I| = |I'|$ and $|J| = |J'|$.

We may thus denote $B_{\kappa,\delta}$ to be the unique (up to isomorphism) rectangular band with κ $\mathcal{R}$-classes and δ $\mathcal{L}$-classes.

Corollary

Rectangular bands are homogeneous. Moreover any homogeneous band B such that $(B, <) \cong \mathcal{A}_n$ is isomorphic to some $B_{i,j}$, where $i \cdot j = n$.

General bands

- While there exists a classification theorem for general bands, it is far too complex for use. Moreover, no general isomorphism theorem exists, so its usefulness in understanding homogeneous bands is minimal. However a weaker form of the theorem will be of use:

Theorem

Let B be an arbitrary band. Then $Y = S/\mathcal{D}$ is a semilattice and B is a semilattice of rectangular bands B_α (which are the $\mathcal{D}$-classes), that is,

$$B = \bigcup_{\alpha \in Y} B_\alpha \text{ and } B_\alpha B_\beta \subseteq B_{\alpha\beta}.$$

- We therefore understand the *global* structure of any band, but not the local structure.

Substructure of homogeneous bands

Lemma (TQG)

If $B = \bigcup_{\alpha \in Y} B_\alpha$ is a homogeneous band, then:

i) *$Aut\,(B)$ is transitive on B, that is if $e, f \in B$ then there exists $\theta \in Aut(B)$ such that $e\theta = f$;*

ii) *$B_\alpha \cong B_\beta$ for all $\alpha, \beta \in Y$.*

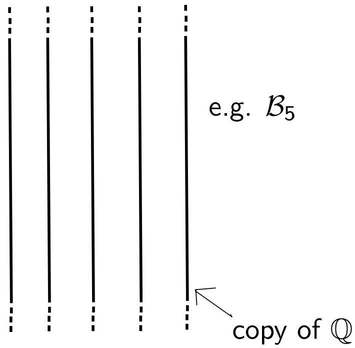
Conjecture

If $B = \bigcup_{\alpha \in Y} B_\alpha$ is a homogeneous then Y is homogeneous.

- However homogeneity does not pass to all induced substructures of B. For example if B is the universal semilattice, then the poset $(B, <)$ is not homogeneous.
- Understanding how the rectangular bands interact in a band is thus key to homogeneity.

Poset 2: $\mathcal{B}_n$

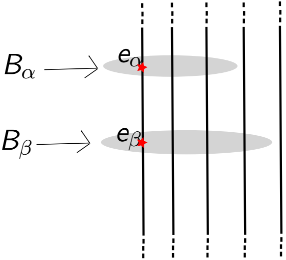
$\mathcal{B}_n = \mathcal{A}_n \times \mathbb{Q}$



The $\mathcal{D}$-classes

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a band with induced poset $\mathcal{B}_n$.

Then if $\alpha > \beta$ and $e_\alpha \in B_\alpha$ then $\exists!$ $e_\beta \in B_\beta$ such that $e_\alpha > e_\beta$.



Bands with induced poset $\mathcal{B}_n$

- Suppose now that $B = \bigcup_{\alpha \in Y} B_\alpha$ is such that $(B, <) \cong \mathcal{B}_n$. Then B satisfies the following condition: for each e_α and $\beta \leq \alpha$, there exists a unique $e_\beta \in B_\beta$ such that $e_\beta < e_\alpha$.

- A **normal band** is a band B satisfying

$$zxyz = zyxz \text{ for all } x, y, z \in B.$$

  This is equivalent to B satisfying the condition above.

- A band B is called a **left/right normal band** if it is normal and each B_α is a left/right-zero band.

The classification for $\mathcal{B}_n$

- Since $\mathcal{B}_n$ can be regarded as $\mathcal{A}_n \times \mathbb{Q}$, it is worth considering bands of the form $B = B_{i,j} \times \mathbb{Q}$.

Lemma (TQG)

The band $B_{i,j} \times Y$ is homogeneous if and only if Y is homogeneous. Moreover $(B_{i,j} \times Y, <)$ is isomorphic to $i \cdot j$ incomparable copies of Y.

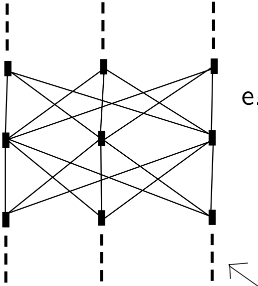
Corollary

A band B is homogeneous and is such that $(B, <) \cong \mathcal{B}_n$ if and only if $B \cong B_{i,j} \times \mathbb{Q}$, where $i \cdot j = n$.

Note: There exists a non-homogeneous normal band with induced poset isomorphic to $\mathcal{B}_n$.

Poset 3: $\mathcal{C}_n$

$$\mathcal{C}_n = \mathcal{A}_n \times \mathbb{Q}, \text{ with } (a,p) < (b,q) \Leftrightarrow p < q.$$

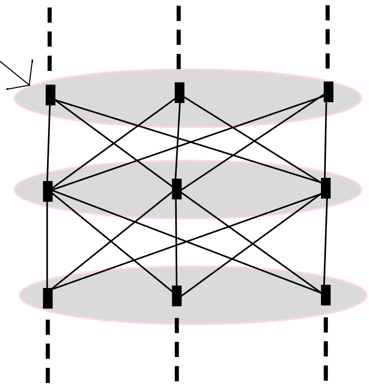


e.g. $\mathcal{C}_3$

copy of $\mathbb{Q}$

The $\mathcal{D}$-classes

$B_\alpha = \{(a, \alpha) : a \in \mathcal{A}_3\}$



Bands with induced poset $\mathcal{C}_n$

Lemma

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a band with $(B, <) \cong \mathcal{C}_n$. Then $Y = \mathbb{Q}$ and each B_α is isomorphic to some $B_{i,j}$, where $i \cdot j = n$. Moreover, the product on B is given by, for any $\alpha > \beta$ in $\mathbb{Q}$,

$$e_\alpha f_\beta = f_\beta = f_\beta e_\alpha.$$

Lemma (TQG)

Let $B = \bigcup_{\alpha \in \mathbb{Q}} B_\alpha$ be a band with $(B, <) \cong \mathcal{C}_n$. Then B is homogeneous if and only if $B_\alpha \cong B_\beta$ for all $\alpha, \beta \in \mathbb{Q}$. Moreover if $C = \bigcup_{\alpha \in \mathbb{Q}} C_\alpha$ is also a homogeneous band with induced poset isomorphic to $\mathcal{C}_n$, then $B \cong C$ if and only if $B_\alpha \cong C_\alpha$.

- We may thus denote $D_{i,j}$ as the unique (up to isomorphism) homogeneous band with induced poset isomorphic to $\mathcal{C}_{i \cdot j}$ and $\mathcal{D}$-classes isomorphic to $B_{i,j}$.

Final case: $\mathbb{P}$

Lemma (TQG)

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a homogeneous band, where $Y \ncong \mathbb{Q}$. Then B is normal.

Lemma

If B is normal then $(B, <) \ncong \mathbb{P}$.

Proof.

Let $e_\alpha, f_\alpha \in B_\alpha$ (so that $e_\alpha \perp f_\alpha$). Then $\nexists g \in B$ such that $g > e_\alpha, f_\alpha$ as B is normal. However in $\mathbb{P}$ every pair of elements has a cover. $\square$

Summary and (possible) classification

Corollary

If P is a homogeneous poset then there exists a homogeneous band B such that $(B, <) \cong P$ if and only if $P \ncong \mathbb{P}$.

Note: Given a homogeneous poset $P \ncong \mathbb{P}$, the existence of a homogeneous band B such that $(B, <) \cong P$ is not unique in general. In fact B is unique up to isomorphism if and only if P is trivial or $(\mathbb{Q}, <)$.

Proposition

The following bands are homogeneous:

i) *Normal type:* $B_{n,m} \times Y$, $B_{LN} \bowtie (B_{1,m} \times Y), (B_{n,1} \times Y) \bowtie B_{RN}$ *or* $B_N = B_{LN} \bowtie B_{RN}$;

ii) *Covering type:* $D_{n,m}, R_{n,m}$ *or* $L_{n,m}$,

for $n, m \in \mathbb{N}^*$ *and* Y *is a homogeneous semilattice (where* B_N, B_{LN}, B_{RN} *is the universal normal/left normal/ right normal band, respectively).*
Moreover if B *is homogeneous, then* B *is normal or one of the covering types above.*