The push forward construction and the Baer sum of special Schreier extensions of monoids

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Special Schreier extensions

Definition

A surjective monoid homomorphism $f\colon A \to B$ is a special Schreier extension if the kernel congruence

$$\begin{tikzcd} Eq(f) \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & A \arrow[l, "\langle 1,1\rangle" description] \end{tikzcd}$$

is a Schreier congruence.

In other terms, if $f(a_1) = f(a_2)$, then there exists a unique $x \in \mathrm{Ker}(f)$ such that $a_2 = x \cdot a_1$.

Special Schreier extensions with abelian kernel

Using the equivalence between Schreier split extensions and monoid actions, we get an action of A on $X = \mathrm{Ker}(f) = \mathrm{Ker}(f_1)$:

$$
\begin{tikzcd}
X \arrow[r, shift right, "q"'] & \mathrm{Eq}(f) \arrow[l, dotted, shift right] \arrow[r, shift left=2, "f_1"] \arrow[r, shift right=2, "f_2"'] & A \arrow[l, "\langle 1,1 \rangle" description] \arrow[r, "f"] & B
\end{tikzcd}
$$

$a \bullet x = q(a \cdot x, a)$.
When X is abelian, this induces an action of B on X:

$$b \bullet x = a \bullet x \quad \text{for } a \in A \text{ such that } f(a) = b.$$

So we can make a partition

$$\mathrm{SchExt}(B, X) = \coprod_{\varphi} \mathrm{SchExt}(B, X, \varphi).$$

Factor sets

Definition

Given a monoid B, an abelian group X and an action $\varphi\colon B \to End(X)$ of B on X, a ***factor set*** *is a map $g\colon B \times B \to X$ which satisfies, for all $b, b_1, b_2, b_3 \in B$, the following conditions:*

(i) $g(b, 1) = g(1, b) = 1$;

(ii) $g(b_1, b_2) \cdot g(b_1 \cdot b_2, b_3) = (b_1 \bullet g(b_2, b_3)) \cdot g(b_1, b_2 \cdot b_3)$.

Given a special Schreier extension with abelian kernel

$$X \xrightarrow{k} A \xrightarrow{f} B,$$

we can associate with it a factor set in the following way: let $s\colon B \to A$ be a set-theoretical section of f such that $s(1) = 1$. Then we get a factor set g by putting

$$g(b_1, b_2) = q(s(b_1) \cdot s(b_2), s(b_1 \cdot b_2))$$

Special Schreier extensions and factor sets

Proposition

Any special Schreier extension of B by X is isomorphic to an extension of the form

$$X \xrightarrow{\langle 1,0 \rangle} X \times B \xrightarrow{\pi_B} B,$$

where the monoid operation on $X \times B$ is defined by:

$$(x_1, b_1) \cdot (x_2, b_2) = (x_1 \cdot (b_1 \bullet x_2) \cdot g(b_1, b_2), b_1 \cdot b_2).$$

Choosing two different sections for f, the corresponding factor sets differ by an **inner factor set**.

The Baer sum of special Schreier extensions

Definition

A factor set g is an ***inner factor set*** *if it is of the form*

$$g(b_1, b_2) = h(b_1) \cdot (b_1 \bullet h(b_2)) \cdot h(b_1 \cdot b_2)^{-1}$$

for some map $h\colon B \to X$ such that $h(1) = 1$.

Theorem

$SchExt(B, X, \varphi)$ is in bijection with the factor abelian group

$$\frac{\mathcal{F}(B, X, \varphi)}{\mathcal{IF}(B, X, \varphi)}.$$

Hence $\mathrm{SchExt}(B, X, \varphi)$ inherits an abelian group structure.

The push forward of a special Schreier extension

Given a special Schreier extension with abelian kernel

$$X \overset{k}{\rightarrowtriangle} A \overset{f}{\twoheadrightarrow} B,$$

inducing the action $\varphi\colon B \to \mathrm{End}(X)$, an action $\psi\colon B \to \mathrm{End}(Y)$ on an abelian group Y, and an equivariant homomorphism $g\colon X \to Y$:

$$g(b \bullet x) = b \bullet g(x)$$

we want to build a universal special Schreier extension of B by Y:

$$\begin{tikzcd} X \arrow[r, tail, "k"] \arrow[d, "g"'] & A \arrow[r, two heads, "f"] & B \\ Y & & \end{tikzcd}$$

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we want to build a universal special Schreier extension of B by Y:

$$\begin{tikzcd} X \arrow[r, "k", tail] \arrow[d, "g"'] & A \arrow[r, "f", two heads] \arrow[d, "g'"'] & B \arrow[d, equal] \\ Y \arrow[r, "k'", tail] & C \arrow[r, "f'", two heads] & B \end{tikzcd}$$

The push forward of a special Schreier extension

Given a special Schreier extension with abelian kernel

$$X \overset{k}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B,$$

inducing the action $\varphi \colon B \to \mathrm{End}(X)$, an action $\psi \colon B \to \mathrm{End}(Y)$ on an abelian group Y, and an equivariant homomorphism $g \colon X \to Y$:

$$g(b \bullet x) = b \bullet g(x)$$

we want to build a universal special Schreier extension of B by Y:

$$\begin{tikzcd}
X \arrow[r, tail, "k"] \arrow[d, "g"'] \arrow[dd, bend left, "u"] & A \arrow[r, two heads, "f"] \arrow[d, "g'"'] \arrow[dd, bend left, "v"] & B \arrow[d, equal] \\
Y \arrow[r, tail, "k'"] \arrow[d, "r"'] & C \arrow[r, two heads, "f'"] & B \arrow[d, equal] \\
Z \arrow[r, tail, "l"'] & E \arrow[r, two heads, "p"'] & B
\end{tikzcd}$$

The push forward of a special Schreier extension

Given a special Schreier extension with abelian kernel

$$\begin{tikzcd} X \arrow[r, tail, "k"] & A \arrow[r, two heads, "f"] & B, \end{tikzcd}$$

inducing the action $\varphi\colon B \to \mathrm{End}(X)$, an action $\psi\colon B \to \mathrm{End}(Y)$ on an abelian group Y, and an equivariant homomorphism $g\colon X \to Y$:

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Y \arrow[r, tail, "k'"] \arrow[d, "r"'] & C \arrow[r, two heads, "f'"] \arrow[d, dotted, "\alpha"'] & B \arrow[d, equal] \\
Z \arrow[r, tail, "l"'] & E \arrow[r, two heads, "p"'] & B
\end{tikzcd}$$

The push forward construction

There is an induced action $\zeta = \psi f : A \to \mathrm{End}(Y)$. The semidirect product

$$Y \xrightarrow{\langle 1,0\rangle} Y \rtimes_\zeta A \underset{\pi_A}{\overset{\langle 0,1\rangle}{\leftrightarrows}} A.$$

is a special Schreier extension. There is a monomorphism

$$h \colon X \to Y \rtimes_\zeta A, \qquad h(x) = (g(x)^{-1}, k(x)).$$

The congruence on $Y \rtimes_\zeta A$ generated by $h(X)$ is given by

$$(y_1, a_1)R(y_2, a_2) \text{ if } \exists\ x \in X \text{ such that } (y_2, a_2) = h(x) \cdot (y_1, a_1).$$

The quotient w.r.t. this congruence gives a special Schreier extension

$$X \xrightarrow{h} Y \rtimes_\zeta A \xrightarrow{c} C.$$

The push forward construction

Applying the Nine Lemma to the diagram

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, tail, "h"] & X \arrow[d, tail, "k"] \\
Y \arrow[r, tail, "\langle 1,0 \rangle"] \arrow[d, equal] & Y \rtimes_{\zeta} A \arrow[r, two heads, "\pi_A"] \arrow[d, two heads, "c"] & A \arrow[d, two heads, "f"] \\
Y \arrow[r, "k'"'] & C \arrow[r, "f'"'] & B,
\end{tikzcd}$$

where f' is induced by the universal property of the quotient, we obtain that the lower row is the universal special Schreier extension we were looking for.
This gives the functoriality of $\mathrm{SchExt}(B,-)\colon B\text{-}Mod \to Set$.

The Baer sum revisited

The push forward allows a functorial description of the Baer sum of special Schreier extensions.
Given two special Schreier extensions

$$X \rightarrowtriangle^{k_1} A_1 \xrightarrow{f_1} B \quad \text{and} \quad X \rightarrowtriangle^{k_2} A_2 \xrightarrow{f_2} B$$

with abelian kernel X which induce the same action $\varphi \colon B \to \mathrm{End}(X)$, we first consider their direct product:

$$X \times X \rightarrowtriangle^{k_1 \times k_2} A_1 \times A_2 \xrightarrow{f_1 \times f_2} B \times B$$

and then

The Baer sum revisited

$$
\begin{tikzcd}
X \arrow[r, tail, "k'"] & C \arrow[r, two heads, "f'"] & B \\
X \times X \arrow[u, "m"] \arrow[r, tail, "\langle k_1, k_2 \rangle"] \arrow[d, equal] & P \arrow[u, "c"'] \arrow[r, two heads, "\bar{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[u, equal] \arrow[d, "\Delta_B"] \\
X \times X \arrow[r, tail, "k_1 \times k_2"'] & A_1 \times A_2 \arrow[r, two heads, "f_1 \times f_2"'] & B \times B.
\end{tikzcd}
$$

This gives a functorial description of the abelian group structure on the set $\mathrm{SchExt}(B, X, \varphi)$.