# Decision problems and subgroups in higher dimensional Thompson groups.

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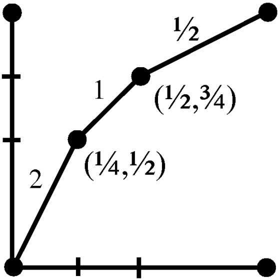
# Subliminal message

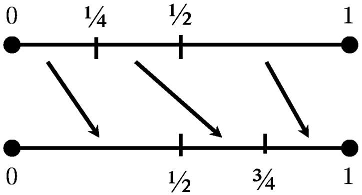
Parabéns Gracinda e Jorge!

# Thompson's group $F$

Thompson's group $F$ is the group $PL_2(I)$, with respect to composition, of all piecewise-linear homeomorphisms of the unit interval $I = [0, 1]$ with a finite number of breakpoints, such that

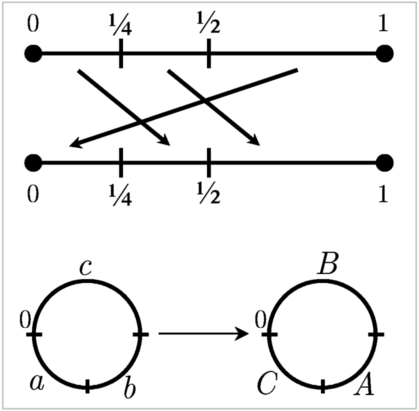
- all breakpoints have dyadic rational coordinates.
- all slopes are integral powers of 2,



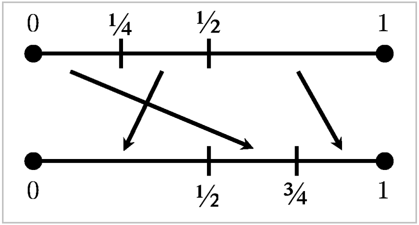
# Thompson's group $T$

Similar to $F$, but with cyclic permutations.



# Thompson's group $V$

Similar to $F$, but with any permutation allowed



## Some results about $F$, $T$, $V$

- $[F, F]$, $T$ and $V$ are infinite simple groups.
- All are f.p. and of type $F_\infty$ (Brown-Geoghegan).
- $V$ contains every finite group as a subgroup.
- $F$ has no free subgroups, but $T$ and $V$ do.
- They all have exponential word growth.

**Question (Open for 50 years)**

*Is $F$ paradoxical or non-paradoxical?*

# Why are Thompson groups interesting?

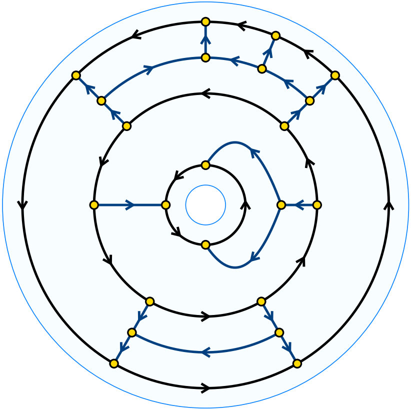
1. Many ways to represent their elements as diagrams.
2. Dynamics helps in the study of these groups.

They show up in many different contexts

- Logic
- Universal algebra and semigroup theory
- Dynamical systems
- Algebraic Topology, Homotopy and $K$-theory
- Computer science (Rotation distance, co-CF groups)
- Analysis: $C^*$-algebras
- Cryptography

**Generalizations:** higher dimensional and braided, diagram groups, Cantor algebra automorphisms, groups acting on fractals.

# Diagrams describe conjugacy and dynamics



Theorem (Belk-M)

*$WP(F, T, V)$ is $O(n)$, while $CP(F, T, V)$ is $O(n^3)$*

# Centralizers in Thompson's group $V$

Theorem (Bleak-Bowman-Gordon-Graham-M-Sapir)

*For any $\alpha \in V$ we have*

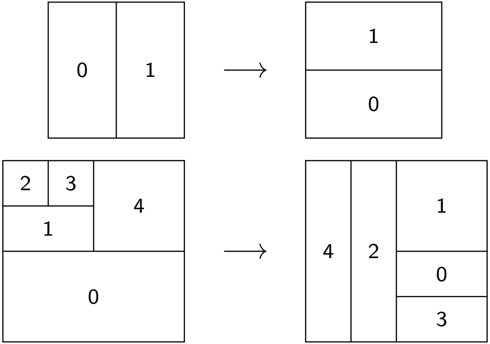
$$C_V(\alpha) \cong \left( \prod_{i=1}^{s} Maps(\mathfrak{C}, C_{n_i}) \rtimes V \right) \times \left( \prod_{j=1}^{t} (A_j \rtimes \mathbb{Z}) \wr Sym(q_j) \right)$$

- *The $Sym(q_j)$'s and $A_k$'s are finite groups relating to the symmetries of "flow graph components".*
- *$Maps(\mathfrak{C}, C_{n_i})$ is the group of continuous maps $\mathfrak{C} \to C_{n_i}$*

# The higher dimensional Thompson groups $nV$

## Definition (Brin)

The group $nV$ is the set of all self-homeomorphisms of $\mathfrak{C}^n$ of the form $h(P_1, P_2)$ where $P_1$ and $P_2$ are numbered patterns with the same number of hypercubes. Locally we have $(x_1, \ldots, x_n) \mapsto (a_1x_1 + b_1, \ldots, a_nx_n + b_n)$, for suitable $a_i$'s and $b_i$'s.



# Are the $nV's$ distinct?

Notice that $\mathfrak{C}^n \simeq \mathfrak{C}$, but is $nV \cong V$?

**Lemma**

*Every element of $V$ has an upper bound on the size of finite orbits.*

**Lemma**

*There is no bound on the finite orbits of the baker's map in $2V$.*

**Theorem (Bleak-Lanoue)**

*$mV \cong nV$ if and only if $m = n$.*

**Theorem (Hennig-M)**

*The groups $nV$ are finitely presented and are simple.*

## Unexpected behavior in $nV$

**Torsion problem** for a f.p. group $G$: given a non-trivial element $g \in G$, can we decide whether or not $g$ has finite order?

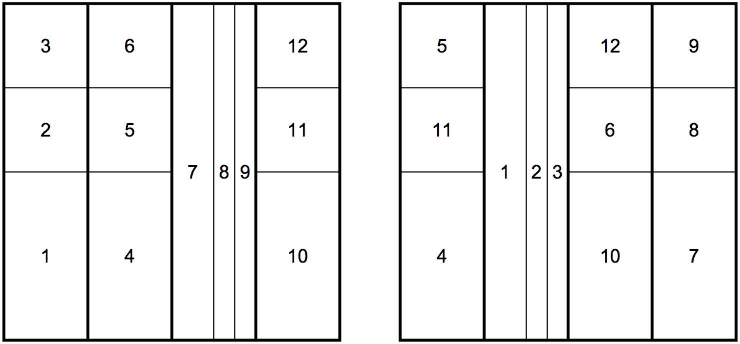
### Lemma (Brin)

*$nV$ has solvable word problem.*

### Theorem (Belk)

*$2V$ has unsolvable torsion problem.*

# Unexpected behavior in $nV$




Theorem (Belk-Bleak)

*Given $f \in 2V$, a dyadic point $p$, and a dyadic rectangle $R$, it is undecidable whether or not the orbit of $p$ intersects $R$.*

# Topologically transitive and mixing elements

**Definition**

1. $f \in 2V$ is **topologically transitive** if for every pair $U, V$ of open sets in $\mathfrak{C}^2$, there is an $n \in \mathbb{Z}$ with $f^n(U) \cap V \neq \emptyset$.

2. $f \in 2V$ is **topologically mixing** if for every pair $U, V$ of open sets in $\mathfrak{C}^2$, there is an $N \in \mathbb{N}$ with $f^n(U) \cap V \neq \emptyset, \forall n \geq N$.

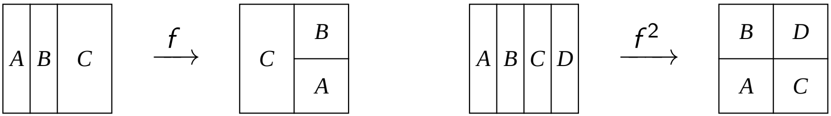
Top. mixing $\implies$ top. transitive (but not the converse).

# Hyperbolic and stretching elements

1. $f \in 2V$ is **stretching** if $f_x(p) > 1$ and $f_y(p) < 1$ $\forall p \in \mathfrak{C}^2$.
2. $f \in 2V$ is **hyperbolic** if $f^n$ is stretching for some $n \in \mathbb{N}$.

**Remark**

*Stretching implies hyperbolic, but the converse does not hold.*

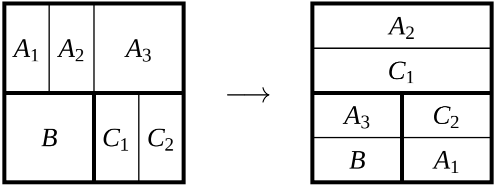



**Proposition**

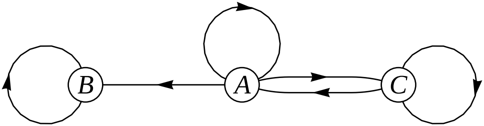
*$S = \{$Stretching elements$\}$ is a semigroup (neither free, nor f.g.) and $2V = \langle S \rangle$ as a group*

# Markov partitions and transition graphs

Hyperbolic elements have nicer partitions (**Markov partitions**)




Such partitions yield a graph containing dynamical information



# Hyperbolic elements

Proposition (Belk-Martínez-M-Nucinkis)

*Let $f \in 2V$ hyperbolic. Then $f$ is topologically conjugate to a two-sided subshift of finite type.*

Theorem (Belk-Martínez-M-Nucinkis)

*Centralizers of hyperbolic elements which are*

1. *topologically transitive with one fixed point, or*
2. *topologically mixing,*

*are virtually cyclic.*

# Subgroups of $nV$

Let $\Gamma$ be a finite graph with vertices $v_1, \ldots, v_n$, no loops and no multiple edges. Then

$$G = \langle g_1, \ldots, g_n \mid g_i g_j = g_j g_i \text{ for all } \{v_i, v_j\} \in \mathrm{Edges}(\Gamma) \rangle$$

is called a **partially commutative group**.

Theorem (Belk-Bleak-M)

*For every $G$ in the following list, there is an $n \in \mathbb{N}$ such that $G \leq nV$:*

1. *Every partially commutative group,*
2. *Every surface group,*
3. *Every f.g. Coxeter group,*
4. *Every 1-relator torsion group,*
5. *Many 3-manifolds groups.*

# Subgroups and undecidability $nV$

**Corollary (Hsu-Wise, weak version)**

*Every partially commutative group can be written as a group of asynchronous automata.*

**Theorem (Belk-Bleak-M)**

*There exists an $n \geq 1$ with the following properties:*

1. *The isomorphism problem for finitely presented subgroups of $nV$ is unsolvable.*
2. *There exists a subgroup $H \leq nV$ that has unsolvable subgroup membership problem and unsolvable conjugacy problem.*

**Theorem (Corwin-Bleak)**

*There is an embedding of $mV$ into $nV$ $\iff$ $m \leq n$.*

# Open questions and more

**Question**

1. *Do surface groups embed into $2V$?*
2. *Wider description of centralizers?*
3. *Conjugacy problem?*

**Fun Fact (Collatz conjecture)**

*There exists an element of $2V$ which replicates the Collatz sequences (If $n$ is even, then $n \mapsto n/2$, if $n$ is odd, then $n \mapsto 3n+1$).*