Circuit Evaluation for Finite Semirings

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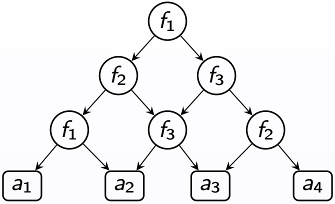
Markus Lohrey
University of Siegen

joint work with Moses Ganardi, Danny Hucke, and Daniel König

Circuits over algebraic structures

$\mathcal{A} = (A, f_1, \dots, f_m), \quad f_i : A^{r_i} \to A$



Circuit $\mathcal{C}$ over $\mathcal{A}$

- set of gates
- output gate
- $X = a \quad (a \in A)$ or $X = f_i(X_1, \dots, X_r)$

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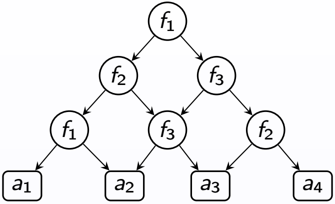

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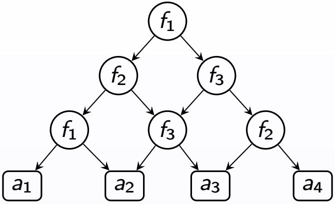

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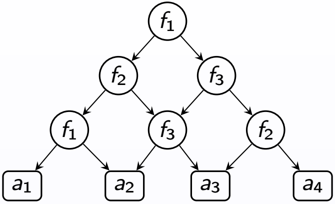

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If $\mathcal{A}$ is finite, then CEP($\mathcal{A}$) is clearly in **P** (= polynomial time).

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Are there structures $\mathcal{A}$ such that $\mathrm{CEP}(\mathcal{A})$ is neither in **NC** nor **P**-complete?

P-complete circuit evulation problems.

Theorem [Ladner, 1975]

Circuit evaluation problem for the boolean semiring $\mathbb{B}_2 = (\{0,1\}, \vee, \wedge)$ is **P**-complete.

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Circuit evaluation problem for the boolean semiring $\mathbb{B}_2 = (\{0, 1\}, \vee, \wedge)$ is $\mathbf{P}$-complete.

A semigroup S is solvable if every group in S is solvable.

Theorem [Beaudry et al., 1993, based on Krohn, Maurer, Rhodes, 1966]

Let $\mathcal{S}$ be a finite semigroup.

- If $\mathcal{S}$ is solvable, then CEP($\mathcal{S}$) is in $\mathbf{NC}$
- otherwise it is $\mathbf{P}$-complete.

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where $A \cdot B = \{ab \mid a \in A,\ b \in B\}$

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The semiring $\mathcal{R} = (R, +, \cdot)$ is $\{0,1\}$-free if it contains no subsemiring with an additive 0 and a multiplicative $1 \neq 0$.

Main Theorem

Theorem

Let $\mathcal{R}$ be a finite semiring.

- If $\mathcal{R}$ is $\{0, 1\}$-free and $(R, \cdot)$ is solvable, then $\mathrm{CEP}(\mathcal{R})$ is in **NC**
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Using results from semigroup theory:

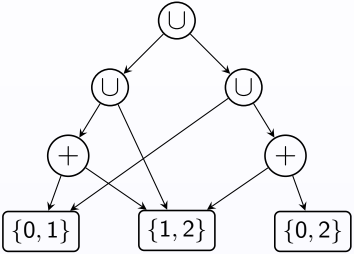
Corollary

Let $\mathcal{S}$ be a finite semigroup.

- If $\mathcal{S}$ is a local group and solvable, then $\mathsf{CEP}(\mathcal{P}(\mathcal{S}))$ is in **NC**
- otherwise it is **P**-complete.

Power semiring over a finite group G

Example: $G = (\mathbb{Z}_5, +)$



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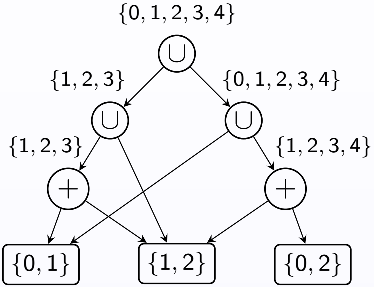

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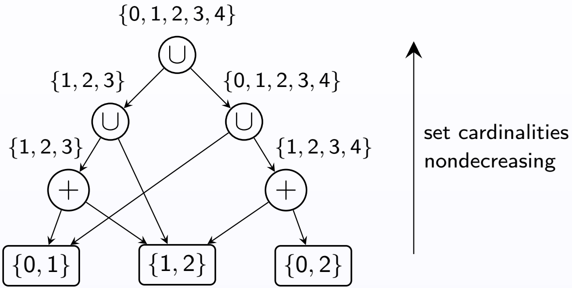

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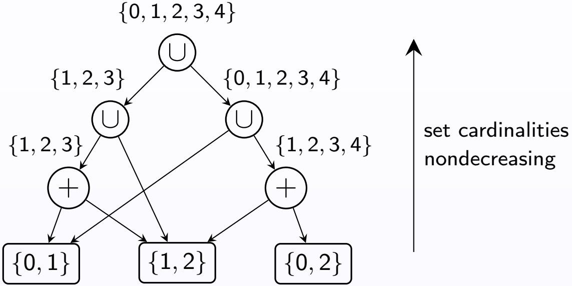


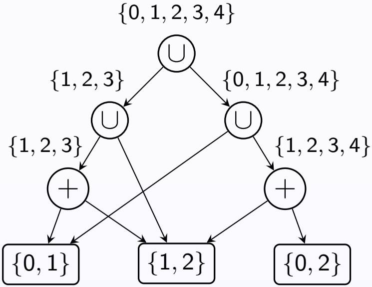
Parallel Evaluation Algorithm

for $k = 1, \ldots, |G|$ **do**

evaluate all gates whose value has size k

endfor

Invariant: After k-th round all sets of size $\leq k$ are evaluated.



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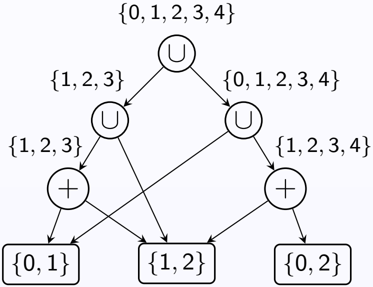

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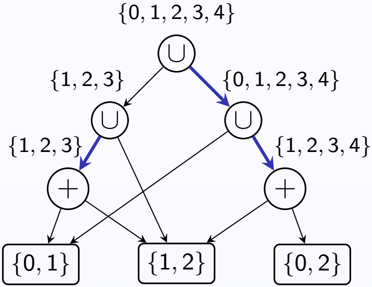
   $\Longrightarrow$ every $\cup$-gate has inner input gate



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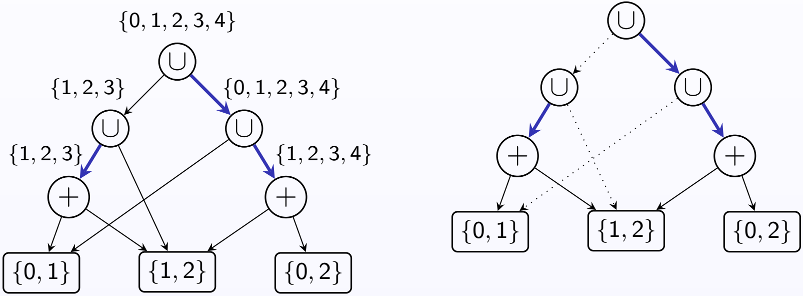
2. $\cup$-gate copies inner input gate



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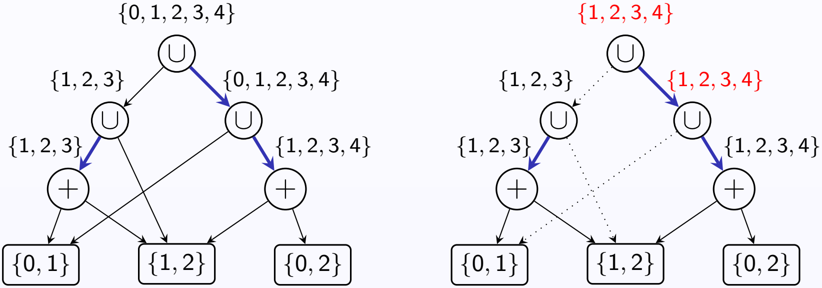

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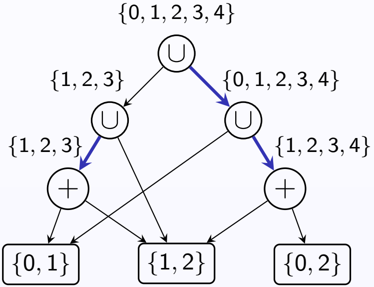
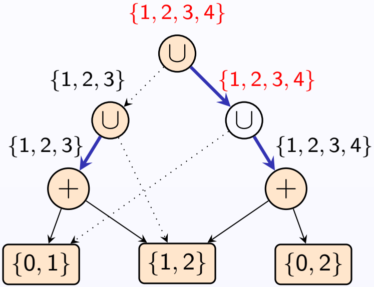
3. Find locally correct gates





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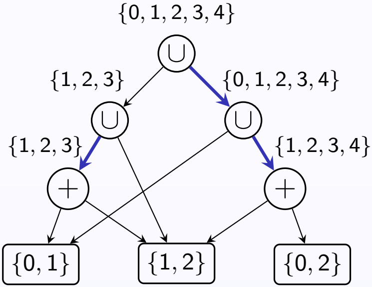

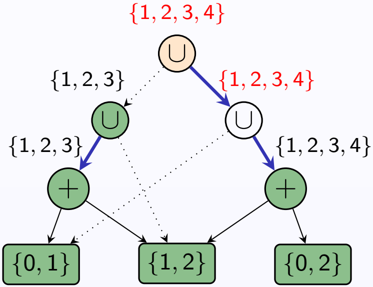

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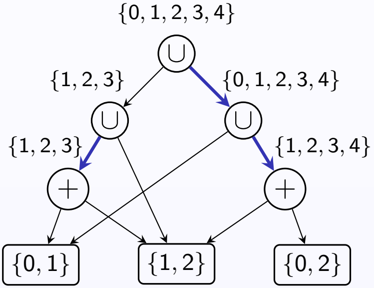

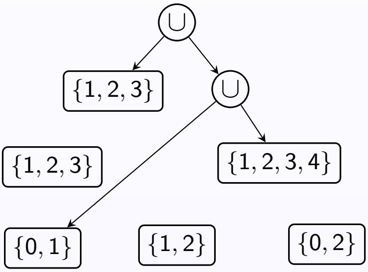

rank-functions

The algorithm terminates after $|R|$ rounds if $\mathcal{R}$ has a function $\mathbf{rank} : R \to \mathbb{N}$ with

- $\mathbf{rank}(a) \leq \mathbf{rank}(a+b)$
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Lemma

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Theorem

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Corollary

If $\mathcal{R}$ is $\{0,1\}$-free and $(R,\cdot)$ is a solvable monoid, then $\mathrm{CEP}(\mathcal{R})$ belongs to **NC**.

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- $\mathcal{H}_e = \mathcal{J}_e \cap eSe$ is the maximal subgroup in S with identity e.

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Then the evaluation $\mathcal{C}$ can be reduced to the evaluation of (a constant number of) circuits with input values from $FSF \setminus \mathcal{H}_e$ (a subsemigroup!).

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Note: eRe is a solvable monoid.

Summary

Theorem

Let $\mathcal{R}$ be a finite semiring.

- If $\mathcal{R}$ is $\{0,1\}$-free and $(R,\cdot)$ is solvable, then $\mathrm{CEP}(\mathcal{R})$ is in **NC** (actually in **DET**).
- otherwise it is **P**-complete.

Outlook

- Intersection problem of a given context-free grammar and a fixed regular language
- Finite "semirings" where $(R,\cdot)$ is a groupoid?
- Evaluating semiring expressions?