New directions in inverse semigroup theory

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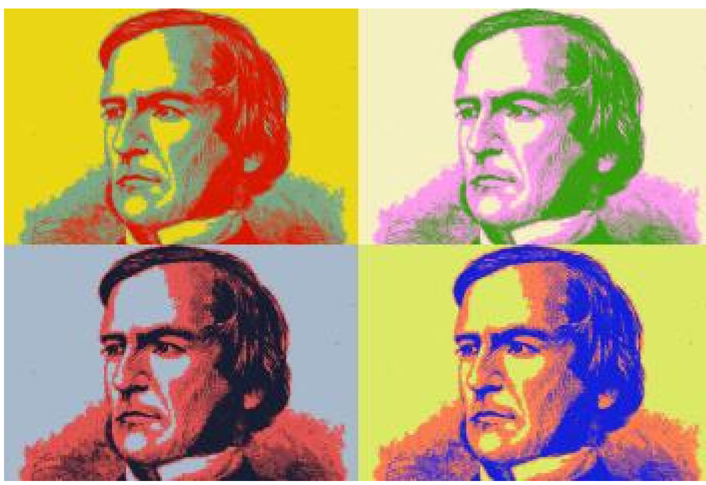

Celebrating the LXth birthday of
Jorge Almeida and Gracinda Gomes

With the collaboration of

Peter Hines, Anja Kudryavtseva, Johannes Kellendonk, Daniel Lenz, Stuart Margolis, Pedro Resende, Phil Scott, Ben Steinberg and Alistair Wallis.

and developing ideas to be found in (amongst others)

Jean Renault, Alexander Kumjian, Alan Paterson, Ruy Exel, Hiroki Matui, . . .

Idea

Develop inverse semigroup theory as the abstract theory of pseudogroups of transformations.

It is worth adding that much if not all 'classical' inverse semigroup theory will be used. For example, fundamental inverse semigroups and the Munn semigroup play an important rôle in this approach.

1. Pseudogroups of transformations

Let X be a topological space. A *pseudogroup of transformations on* X is a collection Γ of homeomorphisms between the open subsets of X (called *partial homeomorphisms*) such that

1. Γ is closed under composition.

2. Γ is closed under 'inverses'.

3. Γ contains all the identity functions on the open subsets.

4. Γ is closed under arbitrary non-empty unions when those unions are partial bijections.

- Pseudogroups are important in the foundations of geometry.

- The idempotents in Γ are precisely the identity functions on the open subsets of the topological space. They form a complete, infinitely distributive lattice or *frame*.

- Johnstone on the origins of frame theory

  > "It was Ehresmann ... and his student Bénabou ... who first took the decisive step in regarding complete Heyting algebras as 'generalized topological spaces'".

  However, Johnstone does not say *why* Ehresmann was led to his frame-theoretic viewpoint of topological spaces. The reason was pseudogroups.

- Pseudogroups are usually replaced by their *groupoids of germs* but pseudogroups nevertheless persist.

- The algebraic part of pseudogroup theory became inverse semigroup theory.

But recent developments show that it is fruitful to bring these divergent approaches back together.

In particular, inverse semigroups and frames.

This is in the spirit of Ehresmann's work.

2. Inverse semigroups

Inverse semigroups arose by abstracting pseudogroups of transformations in the same way that groups arose by abstracting groups of transformations.

There were three independent approaches:

1. Charles Ehresmann (1905–1979) in France.

2. Gordon B. Preston (1925–2015) in the UK.

3. Viktor V. Wagner (1908–1981) in the USSR.

They all three converge on the definition of ‘inverse semigroup’.

Revision

A semigroup S is said to be *inverse* if for each $a \in S$ there exists a unique element a^{-1} such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

The idempotents in an inverse semigroup commute with each other. We speak of the *semilattice of idempotents* $E(S)$ of the inverse semigroup S.

Pseudogroups of transformations are inverse semigroups BUT they also have order-completeness properties.

Let S be an inverse semigroup. Define $a \leq b$ if $a = ba^{-1}a$.

The relation $\leq$ is a partial order with respect to which S is a partially ordered semigroup called the *natural partial order*.

Suppose that $a, b \leq c$. Then $ab^{-1} \leq cc^{-1}$ and $a^{-1}b \leq c^{-1}c$. Thus a necessary condition for a and b to have an upper bound is that $a^{-1}b$ and ab^{-1} be idempotent.

Define $a \sim b$ if $a^{-1}b$ and ab^{-1} are idempotent. This is the *compatibility relation*.

A subset is said to be *compatible* if each pair of distinct elements in the set is compatible.

3. Pseudogroups

An inverse semigroup is said to have *finite (resp. infinite) joins* if each finite (resp. arbitrary) compatible subset has a join.

Definition. An inverse monoid is said to be a *pseudogroup* if it has infinite joins and multiplication distributes over such joins.

Theorem [Schein completion] *Let S be an inverse semigroup. Then there is a pseudogroup $\Gamma(S)$ and a map $\gamma\colon S \to \Gamma(S)$ universal for maps to pseudogroups*.

Pseudogroups are the correct abstractions of pseudogroups of transformations, but their order-completeness properties make them look special.

Let S be a pseudogroup. An element $a \in S$ is said to be *finite* if $a \leq \bigvee_{i \in I} a_i$ implies that $a \leq \bigvee_{i \in 1}^{n} a_i$ for some finite subset $\{1, \ldots, n\} \subseteq I$.

Denote by $\mathsf{K}(S)$ the set of finite elements of S.

An inverse semigroup is said to be *distributive* if it has finite joins and multiplication distributes over such joins.

A pseudogroup S is said to be *coherent* if each element of S is a join of finite elements and the set of finite elements forms a distributive inverse semigroup.

Theorem *The category of distributive inverse semigroups is equivalent to the category of coherent pseudogroups.*

A distributive inverse semigroup is said to be *Boolean* if its semilattice of idempotents forms a (generalized) Boolean algebra.

4. Non-commutative frame theory

Commutative	Non-commutative
frame	pseudogroup
distributive lattice	distributive inverse semigroup
Boolean algebra	Boolean inverse semigroup

We are therefore led to view inverse semigroup theory as non-commutative frame theory:

- Meet semilattices $\longrightarrow$ inverse semigroups.

- Frames $\longrightarrow$ pseudogroups.

Remark There is also a connection with *quantales*. This is discussed in P. Resende, Etale groupoids and their quantales, *Adv. Math.* **208** (2007), 147–209.

5. An example

For $n \geq 2$, define the inverse semigroup P_n by the monoid-with-zero presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \colon a_i^{-1} a_j = \delta_{ij} \rangle,$$

the *polycyclic inverse monoid on* n *generators*.

This is an aperiodic inverse monoid. In particular, its group of units is trivial.

Define C_n to be the distributive inverse monoid generated by P_n together with the 'relation'

$$1 = \bigvee_{i=1}^{n} a_i a_i^{-1}.$$

This was done using a bare-hands method in: The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Communications in algebra* **35** (2007), 4068–4087.

In fact, the C_n are countable atomless Boolean inverse monoids called the *Cuntz inverse monoids*, whose groups of units are the Thompson groups $V_2, V_3, \ldots$, respectively.

They are analogues of $\mathcal{O}_n$, the *Cuntz* C^**-algebras*.

Representations of the inverse monoids C_n are (unwittingly) the subject of *Iterated function systems and permutation representations of the Cuntz algebra* by O. Bratteli and P. E. T. Jorgensen, AMS, 1999.

What is going on here is that elements of P_n are being *glued together* in suitable ways to yield elements of C_n.

This leads to (many) invertible elements being constructed.

Thus an inverse semigroup together with 'suitable relations' can be used to construct new inverse semigroups with interesting properties.

6. Coverages (sketch)

- Informally, a *coverage* $\mathcal{T}$ on an inverse semigroup S is a collection of 'abstract relations' interpreted to mean $a = \bigvee_{i \in I} a_i$.

- **Pseudogroup = inverse semigroup + coverage.**

- If the pseduogroup is coherent, we actually have **distributive inverse semigroup = inverse semigroup + coverage.**

- In special cases, we may have **Boolean inverse semigroup = inverse semigroup + coverage.**

7. Non-commutative Stone dualities

The classical theory of pseudogroups of transformations requires topology.

We generalize the classical connection between topological spaces and frames, which we now recall.

To each topological space X there is the associated frame of open sets $\Omega(X)$.

To each frame L there is the associated topological space of completely prime filters $\mathsf{Sp}(L)$.

The following is classical.

Theorem *The functor $L \mapsto$ Sp(L) from the dual of the category of frames to the category of spaces is right adjoint to the functor $X \mapsto \Omega(X)$.*

A frame is called *spatial* if elements can be distinguished by means of completely prime filters.

A space is called *sober* if points and completely prime filters are in bijective correspondence.

We replace topological spaces (commutative) by topological groupoids (non-commutative).

We view categories as 1-sorted structures (over sets): everything is an arrow. Objects are identified with identity arrows.

A *groupoid* is a category in which every arrow is invertible.

We regard groupoids as 'groups with many identities'. The set of identities is G_o.

Key definition. Let G be a groupoid with set of identities G_o. A subset $A \subseteq G$ is called a *local bisection* if $A^{-1}A, AA^{-1} \subseteq G_o$. We say that S is a *bisection* if $A^{-1}A = AA^{-1} = G_o$.

Proposition *The set of all local bisections of a groupoid forms a Boolean inverse meet-monoid.*

A topological groupoid is said to be *étale* if its domain and range maps are local homeomorphisms.

Why étale? This is explained by the following result.

Theorem [Resende] *A topological groupoid is étale if and only if its set of open subsets forms a monoid under multiplication of subsets.*

Etale groupoids therefore have a strong algebraic character.

There are two basic constructions.

- Let G be an étale groupoid. Denote by $\mathsf{B}(G)$ the set of all open local bisections of G. Then $\mathsf{B}(G)$ is a pseudogroup.

- Let S be a pseudogroup. Denote by $\mathsf{G}(S)$ the set of all *completely prime filters* of S. Then $\mathsf{G}(S)$ is an étale groupoid. [This is the 'hard' direction].

Denote by $\mathbf{Ps}$ a suitable category of pseudogroups and by **Etale** a suitable category of étale groupoids.

Theorem [The main adjunction] *The functor* $\mathsf{G}\colon \mathbf{Ps}^{op} \to \mathbf{Etale}$ *is right adjoint to the functor* $\mathsf{B}\colon \mathbf{Etale} \to \mathbf{Ps}^{op}$*.*

Theorem [The main equivalence] *There is a dual equivalence between the category of spatial pseudogroups and the category of sober étale groupoids.*

We now restrict to coherent pseudogroups. These are automatically spatial.

An étale groupoid is said to be *spectral* if its identity space is sober, has a basis of compact-open sets and if the intersection of any two such compact-open sets is compact-open. We refer to *spectral groupoids* rather than *spectral étale groupoids*.

Theorem *There is a dual equivalence between the category of distributive inverse semigroups and the category of spectral groupoids.*

- Under this duality, a spectral groupoid G is mapped to the set of all *compact-open local bisections* $\mathsf{KB}(G)$

- Under this duality, a distributive inverse semigroup is mapped to the set of all *prime filters* $\mathsf{G}_P(S)$.

An étale groupoid is said to be *Boolean* if its identity space is Hausdorff, locally compact and has a basis of clopen sets. We refer to *Boolean groupoids* rather than *Boolean étale groupoids*.

Proposition *A distributive inverse semigroup is Boolean if and only if prime filters and ultrafilters are the same.*

Theorem *There is a dual equivalence between the category of Boolean inverse semigroups and the category of Boolean groupoids.*

Theorem *There is a dual equivalence between the category of Boolean inverse* meet-semigroups *and the category of* Haudorff *Boolean groupoids.*

Algebra	**Topology**
Semigroup	Locally compact
Monoid	Compact
Meet-semigroup	Hausdorff

The theory discussed so far can be found in: M. V. Lawson, D.H. Lenz, Pseudogroups and their étale groupoids, *Adv. Math.* **244** (2013), 117–170.

8. Boolean inverse semigroups

These have proved to be the most interesting class of inverse semigroups viewed from this perspective. They are genuine non-commutative generalizations of Boolean algebras.

Example Let

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots$$

be a sequence of Boolean inverse $\wedge$-monoids and injective morphisms. Then the direct limit $\varinjlim S_i$ is a Boolean inverse $\wedge$-monoid. If the S_i are finite direct products of finite symmetric inverse monoids then the direct limit is called an *AF inverse monoid*. There is a close connection between such inverse monoids and *MV algebras* (another generalization of Boolean algebras).

Let S be an inverse semigroup. An *ideal* I in S is a non-empty subset such that $SIS \subseteq I$.

Now let S be a Boolean inverse semigroup. A $\vee$-*ideal* I in S is an ideal with the additional property that it is closed under finite compatible joins.

A Boolean inverse semigroup is said to be 0-*simplifying* if it has no non-trivial $\vee$-ideals.

Key definition. A Boolean inverse semigroup that is both fundamental and 0-simplifying is said to be *simple*.

Caution! Simple in this context means what is defined above.

Theorem

1. *The finite Boolean inverse semigroups are isomorphic to the set of all local bisections of a finite discrete groupoid.*

2. *The finite fundamental Boolean inverse semigroups are isomorphic to finite direct products of finite symmetric inverse monoids.*

3. *The finite simple Boolean inverse semigroups are isomorphic to finite direct products of finite symmetric inverse monoids.*

Theorem [The simple alternative] *A simple Boolean inverse monoid is either isomorphic to a finite symmetric inverse monoid or atomless.*

Under classical Stone duality, the Cantor space corresponds to the (unique) countable atomless Boolean algebra; it is convenient to give this a name and we shall refer to it as the *Tarski* algebra.

Corollary *A simple countable Boolean inverse monoid has the Tarski algebra as its set of idempotents.*

A non-zero element a in an inverse semigroup is said to be an *infinitesimal* if $a^2 = 0$. The following result explains why infinitesimals are important.

Proposition *Let S be a Boolean inverse monoid and let a be an infinitesimal. Then*

$$a \vee a^{-1} \vee \overline{(a^{-1}a \vee aa^{-1})}$$

is an involution.

We call an involution that arises in this way a *transposition*.

Programme

The subgroup of the group of units of a simple Boolean inverse monoid generated by the transpositions should be regarded as a generalization of a finite symmetric group. This is the theme of ongoing work by Hiroki Matui and Volodymyr Nekrashevych.

Theorem [Wehrung] *Boolean inverse semigroups form a variety with respect to a suitable signature; this variety is congruence-permutable.*

Programme

Study varieties of Boolean inverse semigroups.

Theorem [Paterson, Wehrung] *Let S be an inverse subsemigroup of the multiplicative semigroup of a C^*-algebra R in such as way that the inverse in S is the involution of R. Then there is a Boolean inverse semigroup B such that $S \subseteq B \subseteq R$ such that the inverse in B is the involution in R.*

Programme

Investigate the relationship between Boolean inverse monoids and C^*-algebras of *real rank zero*. The Cuntz inverse monoids C_n and the AF inverse monoids are good examples.

9. Refinements

Recall that every groupoid is a union of its connected components.

A subset of a groupoid that is a union of connected components is said to be *invariant*.

An étale groupoid is said to be *minimal* if there are no non-trivial open invariant subsets.

Theorem *Under non-commutative Stone duality,* 0*-simplifying Boolean inverse semigroups correspond to minimal Boolean groupoids*.

Let G be an étale groupoid.

Its *isotropy subgroupoid* $\mathrm{Iso}(G)$ is the subgroupoid consisting of the union of its local groups.

The groupoid G is said to be *effective* if the interior of $\mathrm{Iso}(G)$, denoted by $\mathrm{Iso}(G)^{\circ}$, is equal to the space of identities of G.

Let S be an inverse semigroup. It is *fundamental* if the only elements that commute with all idempotents are idempotents.

Theorem *Under the dual equivalences.*

1. *Fundamental spatial pseudogroups correspond to effective sober étale groupoids.*

2. *Fundamental distributive inverse semigroups correspond to effective spectral groupoids.*

3. *Fundamental Boolean inverse semigroups correspond to effective Boolean groupoids.*

Definition Denote by Homeo($\mathscr{S}$) the group of homeomorphisms of the Boolean space $\mathscr{S}$. By a *Boolean full group*, we mean a subgroup G of Homeo($\mathscr{S}$) satisfying the following condition: let $\{e_1, \ldots, e_n\}$ be a finite partition of $\mathscr{S}$ by clopen sets and let $g_1, \ldots, g_n$ be a finite subset of G such that $\{g_1 e_1, \ldots, g_n e_n\}$ is a partition of $\mathscr{S}$ also by clopen sets. Then the union of the partial bijections $(g_1 \mid e_1), \ldots, (g_n \mid e_n)$ is an element of G. We call this property *fullness* and term *full* those subgroups of Homeo($\mathscr{S}$) that satisfy this property.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups.*

2. *Simple Boolean inverse monoids*

3. *Minimal, effective Boolean groupoids.*

Let $\mathscr{S}$ be a compact Hausdorff space. If $\alpha \in$ Homeo($\mathscr{S}$), define

$$\text{supp}(\alpha) = \text{cl}\{x \in \mathscr{S} : \alpha(x) \neq x\}$$

the *support* of α.

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups in which each element has clopen support.*

2. *Simple Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff Boolean groupoids.*

Theorem *The following three classes of structure are equivalent.*

1. *Minimal Boolean full groups in which each element has a* clopen fixed-point set*.*

2. *Simple basic Boolean inverse meet-monoids*

3. *Minimal, effective, Hausdorff, principal Boolean groupoids.*

Let S be a Boolean inverse monoid. Denote by Sym(S) the subgroup of the group of units of S generated by transpositions.

Let a and b be infinitesimals and put $c = (ba)^{-1}$ as in the following diagram

$$\begin{tikzcd} & e_2 \arrow[dl, "b"'] & \\ e_3 \arrow[rr, "c"'] & & e_1 \arrow[ul, "a"'] \end{tikzcd}$$

where the idempotents e_1, e_2, e_3 are mutually orthogonal. Put $e = e_1 \vee e_2 \vee e_3$. Then $a \vee b \vee c \vee \bar{e}$ is a unit called a 3-*cycle*.

Denote by Alt(S) the subgroup of the group of units of S generated by 3-cycles.

Theorem [Nekrashevych] *Let S be a simple Boolean inverse monoid. Then* Alt(S) *is simple.*