Irreducible representations of Chinese monoids

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Conference on Semigroups and Automata
Lisbon, June, 2016

Plan of the talk

For an integer $n \geq 1$ the finitely presented monoid C_n defined by generators $a_1, \ldots, a_n$ and relations

$$a_j a_k a_i = a_k a_j a_i = a_k a_i a_j \qquad \text{for } i \leq j \leq k$$

is called the Chinese monoid of rank n.

If K is a field then $K[C_n]$ denotes the corresponding semigroup algebra, called the Chinese algebra of rank n.

For example, $K[C_1]$ is the polynomial algebra in one variable. Whereas

$$K[C_2] = K\langle x, y \mid yx \text{ is central}\rangle.$$

Elements of C_n admit a normal form. Namely, we have

Theorem (Cassaigne, Espie, Hivert, Krob, Novelli)

Every element $w \in C_n$ can be uniquely written as $w = w_1 \cdots w_n$, where

$$\begin{aligned} w_1 &= a_1^{k_{1,1}}, \\ w_2 &= (a_2 a_1)^{k_{2,1}} a_2^{k_{2,2}}, \\ w_3 &= (a_3 a_1)^{k_{3,1}} (a_3 a_2)^{k_{3,2}} a_3^{k_{3,3}}, \\ &\vdots \\ w_n &= (a_n a_1)^{k_{n,1}} (a_n a_2)^{k_{n,2}} \cdots (a_n a_{n-1})^{k_{n,n-1}} a_n^{k_{n,n}} \end{aligned}$$

and $k_{i,j}$ are non-negative integers.

The elements $w_1, \ldots, w_n$ are know as the Chinese columns. Due to its form elements of C_n correspond to some diagrams (the so-called Chinese staircase).

Let K be a field. Let X be a finitely generated free monoid and $K[X]$ the corresponding free (associative) algebra.

A K-algebra A is finitely presented if it is of the form $K[X]/I$, where I is a finitely generated ideal of $K[X]$.

A special class — algebras defined by homogeneous semigroup relations. This is the case when I is generated by a set of the form

$$\{u - v : (u, v) \in R\}$$

for some $R \subseteq X \times X$ satisfying $|u| = |v|$ for each $(u, v) \in R$.

In this case we have $A \cong K[X/\rho]$, where ρ is the congruence on X generated by the set R.

There are many important (and unsolved in full generality) questions involving finitely presented algebras.

For example, assume that A is such an algebra over a field K.

- Is the Jacobson radical $J(A)$ of A locally nilpotent? (Amitsur)
- If A is a nil algebra, is A necessarily nilpotent, and in consequence finite-dimensional? (Latyshev, Zelmanov)
- Does A (satisfying some additional conditions) have only finitely many minimal prime ideals?

Positive answers to these questions are known only for certain classes of algebras. In particular, the answer to these questions is 'Yes' if A is PI.

What about algebras defined by homogeneous semigroup relations?

Even for this very special class of finitely presented algebras the above mentioned problems seem to be very difficult to solve.

Conclusion: Any (even a partial) result is interesting.

Motivations — plactic algebras

For an integer $n \geq 1$ the finitely presented monoid M_n defined by generators $a_1, \ldots, a_n$ and relations

$$\begin{aligned} a_i a_k a_j &= a_k a_i a_j \qquad \text{for } i \leq j < k, \\ a_j a_i a_k &= a_j a_k a_i \qquad \text{for } i < j \leq k \end{aligned}$$

is called the plactic monoid of rank n.

If K is a field then the corresponding semigroup algebra $K[M_n]$ is called the plactic algebra of rank n. For $n \leq 2$, the algebras $K[M_n]$ and $K[C_n]$ coincide.

In case K is algebraically closed and uncountable, the full description of irreducible representations of $K[M_3]$ (with corresponding primitive ideals) is known. In particular, all such representations turn out to be monomial. It is also known that $J(K[M_3]) = 0$ (Kubat, Okniński).

The structure and representations of $K[M_4]$ seem to be much more complex.

Recently, certain concrete families of irreducible representations of $K[M_4]$ were constructed. Some of them turned out not to be monomial!

It was shown that the Jacobson radical $J(K[M_4]) \neq 0$ of $K[M_4]$ is nilpotent.

It was also shown that the congruence ρ on M_4 determined by $J(K[M_4])$ is non-trivial, and coincides with the intersection of the congruences determined by primitive ideals of $K[M_4]$ corresponding to the constructed simple modules (Cedó, Kubat, Okniński).

Theorem (Cedó, Jaszuńska, Okniński)

Let $K[C_n]$ be the Chinese algebra of rank $n \geq 1$ over a field K.

(1) Every minimal prime ideal of $K[C_n]$ is finitely generated by elements of the form $u - v$, where $u, v \in C_n$ and $|u| = |v| = 2$ or 3.

(2) The minimal prime spectrum $\mathcal{P}$ of $K[C_n]$ is finite. Moreover, for every $P \in \mathcal{P}$ we have $K[C_n]/P \cong K[C_n/\rho_P]$, where ρ_P is the congruence on C_n determined by P. In particular, the algebra $K[C_n]/P$ is again defined by homogeneous semigroup relations.

(3) The Jacobson radical $J(K[C_n])$ of $K[C_n]$ is nilpotent.

Application: The Chinese monoid can be presented as a subdirect product

$$C_n \hookrightarrow \prod_{P \in \mathcal{P}} C_n/\rho_P \hookrightarrow \mathbb{B}^k \times \mathbb{Z}^l \quad \text{for some } k, l \geq 1,$$

where $\mathbb{B} = \langle p, q \mid qp = 1 \rangle$ denotes the bicyclic monoid. Hence, we get a completely new representation of C_n!

If ρ is a congruence on C_n then I_ρ denotes the ideal of $K[C_n]$ corresponding to ρ, that is

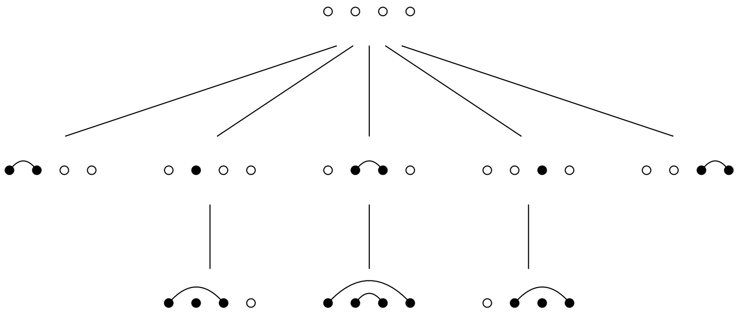
$$I_\rho = \mathrm{Span}_K\{u - v : (u, v) \in \rho\}.$$

A finite tree D is associated to C_n, whose vertices are diagrams of certain special type. Each diagram d in D determines a congruence $\rho(d)$ on C_n in such a way that the ideals $I_{\rho(d)}$ corresponding to the leaves d of D are exactly all the minimal prime ideals of $K[C_n]$.

Each diagram d in D is a graph with n vertices, labeled $1, \ldots, n$ and corresponding to the generators $a_1, \ldots, a_n$ of C_n. For every diagram d that is not the root of D there exist $u, v \in \{1, \ldots, n\}$ satisfying $u \leq v$ such that the vertices $u, \ldots, v$ are marked (colored black) and the corresponding generators $a_u, \ldots, a_v$ are called the used generators in d.

Some pairs k, l (with $k < l$) of the used generators can be connected with an edge and then we say that such a pair is an arc $\widehat{a_l a_k}$ in d. A given generator can be used in at most one arc. The used (marked) generators not appearing in any arc are called dots.

For example, if $n = 4$ then the tree D has the following form



There are five leaves in D, hence the algebra $K[C_4]$ has exactly five minimal prime ideals. Moreover, generators of these ideals can be determined explicitly.

If d_1 is the diagram at level 1 in D with only two used generators that form an arc $\widehat{a_{s+1}a_s}$, where $1 \leq s < n$, then we associate to it the homomorphism $\kappa(d_1)\colon C_n \longrightarrow \overline{C}_n^{s,s+1} \times \mathbb{B} \times \mathbb{Z}$ defined by

$$\kappa(d_1)(a_i) = \begin{cases} (a_i, p, 1) & \text{if } i < s, \\ (1, p, g) & \text{if } i = s, \\ (1, q, 1) & \text{if } i = s+1, \\ (a_i, q, 1) & \text{if } i > s+1, \end{cases}$$

where, for $u, v \in \{1, \ldots, n\}$ satisfying $u \leq v + 1$, we put

$$\overline{C}_n^{u,v} = \langle a_1, \ldots, a_{u-1}, a_{v+1}, \ldots, a_n \rangle / \left(\begin{smallmatrix} a_1, \ldots, a_{u-1} & \text{commute} \\ a_{v+1}, \ldots, a_n & \text{commute} \end{smallmatrix} \right).$$

The congruence $\rho(d_1) = \operatorname{Ker} \kappa(d_1)$ on C_n is then generated by the pairs:

$$\begin{array}{lll} (a_i a_j, a_j a_i), & (a_i a_{s+1} a_j, a_j a_{s+1} a_i) & \text{for } i, j \leq s, \\ (a_k a_l, a_l a_k), & (a_k a_s a_l, a_l a_s a_k) & \text{for } k, l > s. \end{array}$$

Exact rules of construction of diagrams d_t at each level t in the tree D, together with corresponding homomorphisms

$$\kappa(d_t)\colon C_n \longrightarrow \overline{C}_n^{u,v} \times (\mathbb{B} \times \mathbb{Z})^k \times \mathbb{Z}^l$$

and congruences $\rho(d_t) = \operatorname{Ker} \kappa(d_t)$ are described in detail in

- Ł. Kubat and J. Okniński, *Irreducible representations of the Chinese monoid*, `arXiv:1601.00580`.

However, if $t > 1$ then generators of the congruence $\rho(d_t)$ are hard to determine explicitly. Though, if the diagram d_t is of some special shape then looking at the embedding $C_n/\rho(d_t) \longrightarrow \overline{C}_n^{u,v} \times (\mathbb{B} \times \mathbb{Z})^k \times \mathbb{Z}^l$, induced by the homomorphism $\kappa(d_t)$, it is quite easy to derive some relations that must hold in the monoid $C_n/\rho(d_t)$ (relations obtained in this way are crucial in classification of irreducible representations).

Our first result shows that infinite-dimensional representations are crucial.

Proposition

Let $\phi\colon C_n \longrightarrow \mathrm{End}_K(V)$ be an irreducible representation of C_n over a field K. If $\dim_K V < \infty$ then $\phi(C_n)$ is commutative. Hence, if K is algebraically closed then $\dim_K V = 1$.

One of main tools used in classification of irreducible representations of C_n is the following consequence of the Density Theorem

Proposition

Let K be an algebraically closed field. Let A be a left primitive K-algebra such that $\dim_K A < |K|$. Then A is a central K-algebra, that is the center $Z(A)$ of A is (isomorphic to) K.

Therefore, from now we assume that the base field K is algebraically closed and uncountable!

First non-trivial case is $n = 2$. Since $a_2 a_1$ is a central element of $K[C_2]$, it follows that for each simple left $K[C_2]$-module V we have $(a_2 a_1 - \lambda)V = 0$ for some $\lambda \in K$.

If $\lambda = 0$ it is easy to see that in fact $a_i V = 0$ for some $i \in \{1, 2\}$. Hence, V may be treated as a simple module over the algebra $K[C_2]/(a_i) \cong K[C_1]$, and its structure is pretty easy to determine.

However, if $\lambda \neq 0$ then V may be treated as a simple module over the algebra $A = K[C_2]/(a_2 a_1 - \lambda) \cong K[\mathbb{B}]$, and its structure is also well known. In particular, if V is additionally a faithful A-module then $V = \bigoplus_{i \geq 0} K e_i$ with the action of generators given by

$$a_1 e_i = \lambda e_{i+1}, \qquad a_2 e_i = \begin{cases} e_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Our next aim is to 'extend' the above representation of C_2 to arbitrary C_n for n even. It turns out that representations obtained in this way are the corner stone of an inductive classification of all irreducible representations of C_n.

Construction Lemma

Let V be a K-linear space with basis $\{e_{i_1,\ldots,i_s} : i_1,\ldots,i_s \geq 0\}$ for some $s \geq 1$. Moreover, let $0 \neq \lambda_1,\ldots,\lambda_s \in K$ and $n = 2s$. Then the action of generators $a_1,\ldots,a_n \in C_n$ on V defined by

$$a_j e_{i_1,\ldots,i_s} = \begin{cases} \lambda_j e_{i_1,\ldots,i_{j-1},i_j+1,\ldots,i_s+1} & \text{if } j \leq s, \\ e_{i_1,\ldots,i_{n-j},i_{n-j+1}-1,\ldots,i_s-1} & \text{if } j > s \text{ and } i_{n-j+1},\ldots,i_s > 0, \\ 0 & \text{otherwise} \end{cases}$$

makes $V = V(\lambda_1,\ldots,\lambda_s)$ a simple left $K[C_n]$-module. Moreover, if $0 \neq \mu_1,\ldots,\mu_s \in K$ then $V(\lambda_1,\ldots,\lambda_s) \cong V(\mu_1,\ldots,\mu_s)$ as left $K[C_n]$-modules if and only if $\lambda_i = \mu_i$ for all $i = 1,\ldots,s$.

Irreducible representations of C_n — classification

Let V be a simple left $K[C_n]$-module with annihilator P. Since P is a prime ideal, it contains a minimal prime ideal of $K[C_n]$, which is of the form $I_{\rho(d)}$ for some leaf d in the tree D.

So, our strategy is to investigate the structure of left primitive ideals of $K[C_n]$ containing ideals coming from diagrams of a particular shape.

Proposition

Assume that P is a left primitive ideal of $K[C_n]$ containing the ideal I_ρ, where ρ is the congruence on C_n determined by the diagram

consisting of $t > 0$ consecutive arcs $\widehat{a_{s+1}a_s}, \ldots, \widehat{a_{s+t}a_{s-t+1}}$. Assume additionally that $a_{s+t}a_{s-t+1} \in P$. Then $a_{s+t} \in P$ or $a_{s-t+1} \in P$.

There are also similar results in case P contains the ideal I_ρ, where ρ is the congruence on C_n determined by other diagrams of special shapes.
These results lead to a conclusion that $\lambda a_j - \mu a_{j-1} \in P$ for some $j \in \{2, \ldots, n\}$ and some $\lambda, \mu \in K$ not both equal to zero.

Now, we are ready to formulate the main result.

Classification Theorem

Let V be a simple left $K[C_n]$-module. Then V is isomorphic to one of the modules described in the Construction Lemma (in this case n must be even) or $xV = 0$, where $x = a_i - \lambda$ for some $i \in \{1, \ldots, n\}$ and $\lambda \in K$, or $x = \lambda a_j - \mu a_{j-1}$ for some $j \in \{2, \ldots, n\}$ and $\lambda, \mu \in K$ not both equal to zero. In the latter case V may be treated as a simple left $K[C_{n-1}]$-module and its structure can be described inductively.

Sketch of the proof. Let P denote the annihilator of V. Then P contains a minimal prime ideal of $K[C_n]$, which is of the form $I_{\rho(d)}$ for some leaf d in D. We also know that the congruence $\rho(d)$ arises as a finite extension $\rho(d_1) \subseteq \cdots \subseteq \rho(d_m) = \rho(d)$, where each d_j is a diagram at level j of D. In particular, $I_{\rho(d_j)} \subseteq P$ for each $j = 1, \ldots, m$.

The remaining part of the proof depends strongly on the form of relations holding in $C_n/\rho(d_j)$.

First, if in the inductive construction of d a dot appears, or if a dot does not appear, but d contains the arc $\widehat{a_n a_i}$, where $i > 1$ or $\widehat{a_j a_1}$, where $j < n$ then, using mentioned propositions, one can show that $xV = 0$, where $x \in K[C_n]$ has one of forms stated in Theorem.

If n is odd then one of the cases already described must hold (the diagram d contains necessarily a dot or the arc $\widehat{a_n a_i}$, where $i > 1$ or $\widehat{a_j a_1}$, where $j < n$). Therefore, we may assume that $n = 2s$ for some $s \geq 1$. Moreover, it remains to consider the case in which the diagram d

consists of s consecutive arcs $\widehat{a_{s+1} a_s}, \ldots, \widehat{a_n a_1}$.

In the last case it can be proved (using some more technical results) that again $xV = 0$, where $x \in K[C_n]$ has one of forms described in Theorem, or there exist $0 \neq v \in V$ and $0 \neq \lambda_1, \ldots, \lambda_s \in K$ such that the set

$$E = (\lambda_1^{-1} a_{n-1} a_1)^* (\lambda_2^{-1} a_{n-2} a_2)^* \cdots (\lambda_{s-1}^{-1} a_{s+1} a_{s-1})^* (\lambda_s^{-1} a_s)^* v$$

constitutes a basis of V. Moreover, in the latter case, the action of $a_1, \ldots, a_n$ on E agrees with the action of generators defined in the Construction Lemma. Hence, $V \cong V(\lambda_1, \ldots, \lambda_s)$. This ends the proof.

From the proof of the Classification Theorem it also follows that irreducible representations of C_n have particularly transparent form.

Recall that a representation of a monoid M in a K-linear space V is said to be monomial, if V admits a basis E such that for each $w \in M$ and each $e \in E$ there exist $\lambda \in K$ and $f \in E$ such that $we = \lambda f$.

Corollary

Each irreducible representation of the Chinese monoid C_n is monomial.

Thank you
for your attention!