Automaton semigroups: new construction results and examples of non-automaton semigroups

Tara Brough

International Conference on Semigroups and Automata,
Lisbon, 20 June 2016

Joint work with Alan Cain

Automaton semigroups

Automaton semigroup

- An automaton consists of a finite set of states Q, a finite alphabet B of symbols, and a transition relation $\delta : Q \times B \to Q \times B$.
- The set B^* of all strings of symbols from B can be identified with a regular rooted tree of degree $|B|$.
- States in Q act on B^* as tree endomorphisms. This action extends naturally to an action of Q^+ on B^*.
- The subsemigroup of $\mathrm{End}(B^*)$ generated by Q is denoted $\Sigma(\mathcal{A})$.
- A semigroup S is an automaton semigroup if $S \cong \Sigma(\mathcal{A})$ for some automaton $\mathcal{A}$.

Examples and properties

Some examples of automaton semigroups:

- Finite semigroups
- Finitely generated free semigroups of rank greater than 1
- Finitely generated free monoids
- Finitely generated free commutative semigroups (rank > 1)

All automaton semigroups have the following properties:

- They are residually finite: any two distinct elements can be distinguished in some finite quotient.
- They have decidable uniform word problem: there is an algorithm to decide, given automaton $\mathcal{A}$ and $u, v \in Q^+$, whether u and v represent the same element of $\Sigma(\mathcal{A})$.

Non-examples

- The bicyclic monoid $B = \mathrm{Mon}\langle b, c \mid bc = 1\rangle$ is not an automaton semigroup, because it is not residually finite.
- $(\mathbb{N}, +)$, the free semigroup of rank 1, is residually finite, but is not an automaton semigroup (Cain '09).
- Until 2015, $(\mathbb{N}, +)$ was the only proven example of a residually finite non-automaton semigroup.

Open problem: Develop a general method to prove that a semigroup does not arise as an automaton semigroup.

Finding periodic elements

If $p = (p, p, \ldots, p)\tau$, then p acts as a transformation of B and is hence periodic. Generalising this a little:

Lemma

Let $\mathcal{A} = (Q, B, \delta)$ be an automaton such that $\Sigma(\mathcal{A})$ has a zero. If there exist $q, z \in Q$, with z representing the zero element of S, such that q recurses only to itself and z, then q represents a periodic element of S.

Proof.

Let $q = (q_1, \ldots, q_k)\tau$, with $q_i \in \{q, z\}$.
For any n, we have $q^n = (u_{n1}, \ldots, u_{nm})\tau^n$, where each u_{ni} can be expressed as a product of n elements from $\{q, z\}$, and is hence in $\{q^n, z\}$.
This means that two distinct powers of q must have have identical recursion patterns and hence represent the same element of S, and so q is periodic. □

Infinitely many non-examples

Theorem

No nontrivial subsemigroup of $\mathbb{N}^0$ is an automaton semigroup.

Proof strategy:

- $S \leq \mathbb{N}^0$. Assume $S = \Sigma(\mathcal{A})$ for some automaton $\mathcal{A}$.
- Let $Q = \{q_i \mid i \in Y\}$ be the set of states of $\mathcal{A}$, where $Y \subseteq \mathbb{N} \cup \{z\}$.
- Let $k = \max(Y \cap \mathbb{N})$, $\ell = \min(Y \cap \mathbb{N})$. If $k = \ell$ then done.
- Consider $k\ell = q_k^\ell = q_\ell^k$.
- $q_k^\ell = (u_1, \ldots, u_r)\sigma$, $u_i \in Y^\ell$; $q_\ell^k = (v_1, \ldots, v_r)\sigma$, $v_i \in Y^k$.
- $u_i =_S v_i$ for all $i \Rightarrow S$ has periodic elements.
- Hence the automaton $\mathcal{A}$ does not exist.

Automaton semigroup constructions

Questions (Cain 2009):

- If S and T are automaton semigroups, must their free product $S \star T$ be an automaton semigroup? Or perhaps just for S and T finite?
- If S is an automaton monoid and T a finite monoid, is the wreath product $S \wr T$ an automaton monoid?
- Is the class of automaton semigroups closed under taking small extensions?
- If S is not an automaton semigroup, is it possible for S^0 to be an automaton semigroup?
- Rees matrix constructions. Is it possible to characterize completely simple automaton semigroups in terms of automaton groups?

Automaton semigroup constructions

Theorems (TB and Cain 2015):

- If S and T are automaton semigroups, each containing an idempotent, then $S \star T$ is automaton semigroup.
- If S is an automaton monoid and T a finite monoid, then $S \wr T$ an automaton monoid.
- Let S be an automaton semigroup and T a finite semigroup with a right zero. Then any semigroup $S = S \cup T$ having T as an ideal is an automaton semigroup.
- Let S be a Rees matrix semigroup $\mathcal{M}[M; I, \Lambda; P)$ with I and Λ finite, M an automaton monoid and P a $0, 1$-matrix. Then S is an automaton semigroup.
- Strong semilattices.