Cross-connections of linear transformation semigroup

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Regular semigroups

- The talk will be on the theory of cross-connections for regular semigroups with special emphasis on the linear transformation semigroup.
- A semigroup S is said to be (von Neumann) *regular* if for every $a \in S$, there exists b such that $aba = a$.
- In the study of the structure theory of regular semigroups, T E Hall (1973) used the ideals of the regular semigroup to analyse its structure.
- P A Grillet (1974) refined Hall's theory to abstractly characterize the ideals as *regular partially ordered sets* and constructed the fundamental image of the regular semigroup as a cross-connection semigroup.
- In 1994, Nambooripad generalized this idea to any arbitrary regular semigroups by characterizing the ideals as *normal categories*.

Normal categories

- A *normal category* is a categorical abstraction of the principal left(right) ideals of a regular semigroup S.
- So the objects of a normal category of principal left(right) ideals are Se (eS); and the morphisms are partial right(left) translations.
- A normal category $\mathcal{C}$ is axiomatized as a small category with subobjects such that each morphism in $\mathcal{C}$ has a special kind of factorization called normal factorization and each $c \in v\mathcal{C}$ has an associated idempotent normal cone.
- All the *normal cones* in a normal category with a peculiar binary composition forms a regular semigroup $T\mathcal{C}$ known as the semigroup of normal cones in $\mathcal{C}$.
- A *cross-connection* between two normal categories $\mathcal{C}$ and $\mathcal{D}$ is a *local isomorphism* $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ where $N^*\mathcal{C}$ is the *normal dual* of the category $\mathcal{C}$.

Cross-connections

- The normal dual $N^*\mathcal{C}$ is a full subcategory of $\mathcal{C}^*$ where $\mathcal{C}^*$ is the category of all functors from $\mathcal{C}$ to **Set**.
- Hence the objects of $N^*\mathcal{C}$ are functors called $H-$functors and morphisms are natural transformations.
- And given the cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, we have a dual cross-connection $\Delta : \mathcal{C} \to N^*\mathcal{D}$ such that there is a natural isomorphism χ_Γ between the bi-functors $\Gamma(-,-)$ and $\Delta(-,-)$ associated with Γ and Δ.
- Using the natural isomorphism χ_Γ, we can get a *linking* of some normal cones $\gamma \in T\mathcal{C}$ with $\delta \in T\mathcal{D}$.
- And these linked cone pairs (γ, δ) will form a regular semigroup which is called the *cross-connection semigroup* $\tilde{S}\Gamma$ determined by Γ.
- Then $\tilde{S}\Gamma$ is isomorphic to S; and hence giving a faithful representation of the semigroup S as a sub-direct product of $T\mathcal{C} \times (T\mathcal{D})^{\text{op}}$.

Linear transformation semigroup

- Now we proceed to discuss the normal categories arising from the semigroup T_V of singular linear transformations on an arbitrary vectorspace V over a field K.
- T_V is the most important regular subsemigroup of the semigroup $\mathscr{T}_V$ of all(including non-singular) linear transformations on V.
- The cross-connections of $\mathscr{T}_V$ was studied in detail by D Rajendran (cf. [10]) using a different approach.

Lemma 1

If α, β are arbitrary linear transformations on V.

1. $\alpha \mathscr{L} \beta \iff V\alpha = V\beta$.
2. $\alpha \mathscr{R} \beta \iff N_\alpha = N_\beta$.
3. $\alpha \in T_V$ is an idempotent $\iff V = N_\alpha \oplus V\alpha$.

Subspace category

- The proper subspaces of a vectorspace V with linear transformations as morphisms form a category $\mathscr{S}(V)$ called the *subspace category*.
- $\mathscr{S}(V)$ has a natural *choice of subobjects*- the one provided by subspace inclusions.
- Given any linear transformation f between subspaces A and B, then it has a special factorisation of the form $f = quj$ where $q : A \to A'$ is a projection, $u = f_{|A'}$ is an isomorphism and $j = j(B', B)$ is an inclusion.
- Here A' is a complement of the nullspace N_f of f in A and $q : A \to A'$ is the projection associated with the direct sum decomposition $N_f \oplus A' = A$. And $B' = \text{Im } f$.
- Such a factorization is called a *normal factorization* and qu is called the *epimorphic component* f° of f.

Subspace category

- Now given any $D \subseteq V$, we associate a function $\sigma : v\mathscr{S}(V) \to \mathscr{S}(V)$ with the following properties.
  1. For each subspace A of V, $\sigma(A) : A \to D$ and whenever $A \subseteq B$, $j(A, B)\sigma(B) = \sigma(A)$.
  2. For some subspace C of V, $\sigma(C) : C \to D$ is an isomorphism.
- Such a collection of morphisms $\{\sigma(A) \; : \; A \in v\mathscr{S}(V)\}$ is called a *normal cone* σ with vertex D in the category $\mathscr{S}(V)$. In addition if $\sigma(D) = 1_D$, then σ is known as an *idempotent* normal cone.
- Let $u : V \to D$ be a transformation such that $u(x) = x \quad \forall x \in D$.
  For any $A \subseteq V$, define

$$\sigma(A) = u_{|A} : A \to D.$$

Subspace category

- Then σ is an idempotent normal cone with $\sigma(D) = 1_D$ and hence $\mathscr{S}(V)$ is a normal category.
- Now suppose γ, δ are two normal cones in $\mathscr{S}(V)$ with vertices C and D respectively, we can compose them as follows. For any $A \in v\mathscr{S}(V)$,

$$(\gamma * \delta)(A) = \gamma(A)(\delta(C))^\circ \tag{1}$$

  where $(\delta(C))^\circ$ is the epimorphic component of the morphism $\delta(C)$.
- Then it can be seen that $\gamma * \delta$ is a normal cone with vertex D.
- The set of all normal cones in $\mathscr{S}(V)$ under the binary operation defined in equation (1) forms a regular semigroup $T\mathscr{S}(V)$ called the *semigroup of normal cones* in $\mathscr{S}(V)$.

Subspace category

- It can be shown that every normal cone σ in $\mathscr{S}(V)$ defines a linear transformation $\alpha : X \to A$ as follows.
- If B is a basis of V, let

$$(b)\alpha = (b)\sigma(\langle b \rangle) \text{ for all } b \in B \tag{2}$$

where $\sigma(\langle b \rangle)$ is the component of σ at the subspace $\langle b \rangle \in v\mathscr{S}(V)$.

- Conversely every transformation $\alpha : X \to A$ determines a normal cone ρ^α in $\mathscr{S}(V)$ called *principal cone*.
- Thus every normal cone in $\mathscr{S}(V)$ are principal cones and we can further show that

Theorem 2

$\mathscr{S}(V)$ is a normal category and $T\mathscr{S}(V)$ is isomorphic to T_V.

Normal dual

- From $\mathscr{S}(V)$, we construct a new category $N^*\mathscr{S}(V)$ called the *normal dual* of $\mathscr{S}(V)$ whose objects are certain set-valued functors called $H-$functors.
- For an idempotent transformation $e \in T_V$, define a $H-$functor $H(e;-) : \mathscr{S}(V) \to \mathbf{Set}$ as follows. For each $A \in v\mathscr{S}(V)$ and for each $g : A \to B$ in $\mathscr{S}(V)$,

$$H(e;A) = \{ef : f : \text{Im } e \to A\} \text{ and} \tag{3a}$$

$$H(e;g) : H(e;A) \to H(e;B) \text{ given by } ef \mapsto efg. \tag{3b}$$

- It can be shown that the H-functor $H(e;-)$ is determined by the nullspace N_e; and hence inspiring us to define the following category $\mathscr{N}(V^*)$.
- The objects of $\mathscr{N}(V^*)$ are A° where $A^\circ = \{f \in V^* : vf = 0 \text{ for all } v \in A\}$ is the annihilator of A; where A is a subspace of V.

Normal dual

- A morphism in $\mathscr{N}(V^*)$ between A° and B° is abstractly a natural transformation σ between H-functors $H(e;-)$ and $H(f;-)$.
- This σ is determined by a $u \in f(T_V)e$.
- And for a linear map $u : V \to W$, the transpose u^* of u is the linear map from W^* to V^* given by $u^* : \alpha \mapsto u\alpha$ for all $\alpha \in W^*$.
- Hence a morphism in $\mathscr{N}(V^*)$ is given by $u^* : (N_e)^\circ \to (N_f)^\circ$ such that $u \in f(T_V)e$.
- We can see that $\mathscr{N}(V^*)$ is a sub-category of $\mathscr{S}(V^*)$ and is Im P if we define a functor $P : N^*\mathscr{S}(V) \to \mathscr{S}(V^*)$ as

$$vP(H(e;-)) = (N_e)^\circ \quad \text{and} \quad P(\sigma) = u^* \tag{4}$$

where $(N_e)^\circ$ is the annihilator of the nullspace of e and $\sigma(C) : a \mapsto ua$, $u \in f(T_V)e$ and V^* is the algebraic dual space of V.

Normal dual

- If V is finite dimensional, then vP is an order isomorphism; P is $v-$surjective and full. And hence

Theorem 3

Let V be a finite dimensional vectorspace over K, then $N^*\mathscr{S}(V)$ is isomorphic to $\mathscr{S}(V^*)$ as normal categories.

- In general, $N^*\mathscr{S}(V)$ is isomorphic to $\mathscr{N}(V^*)$.
- It can also be shown that $N^*\mathscr{N}(V^*)$ is isomorphic to $\mathscr{S}(V)$.
- Having characterized the normal categories of T_V as $\mathscr{S}(V)$ and $\mathscr{N}(V^*)$, now we proceed to construct some cross-connections between them; and describe the semigroups arising from them.

Cross-connection

Definition 4

Let $\mathcal{C}$ be a small category with subobjects. Then an *ideal* $\langle c \rangle$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ whose objects are subobjects of c in $\mathcal{C}$. It is called the principal ideal generated by c.

Definition 5

Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a *local isomorphism* if F is inclusion preserving, fully faithful and for each $c \in v\mathcal{C}$, $F_{|\langle c \rangle}$ is an isomorphism of the ideal $\langle c \rangle$ onto $\langle F(c) \rangle$.

Definition 6

A *cross-connection* from $\mathcal{D}$ to $\mathcal{C}$ is a triplet $(\mathcal{D}, \mathcal{C}; \Gamma)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a local isomorphism such that for every $c \in v\mathcal{C}$, there is some $d \in v\mathcal{D}$ such that $c \in M\Gamma(d)$.

- The M-set associated with a cone σ in $\mathscr{S}(V)$ (also written $MH(\sigma;-)$) is given by

$$M\sigma = \{c \in \mathscr{S}(V) \mid \sigma(c) \text{ is an isomorphism}\}.$$

- Now in our case, it can be characterized as follows.

Proposition 7

The M-set of the cone ρ^e is given by
$M((N_e)^\circ) = MH(e;-) = M\rho^e = \{A \subseteq V : A \oplus N_e = V\}$.

- From the previous discussion, $(\mathscr{N}(V^*), \mathscr{S}(V), \Gamma)$ is a cross-connection if $\Gamma : \mathscr{N}(V^*) \to \mathscr{N}(V^*)$ is a local isomorphism such that for every $A \in v\mathscr{S}(V)$, there is some $Y \in v\mathscr{N}(V^*)$ such that $A \in M(\Gamma(Y))$.

Associated bi-functors

Now given a cross-connection $\Gamma : \mathscr{N}(V^*) \to \mathscr{N}(V^*)$ with a dual cross-connection $\Delta : \mathscr{S}(V) \to \mathscr{S}(V)$, we have two associated bi-functors $\Gamma : \mathscr{S}(V) \times \mathscr{N}(V^*) \to \textbf{Set}$ and $\Delta : \mathscr{S}(V) \times \mathscr{N}(V^*) \to \textbf{Set}$ such that for all $(A, Y) \in v\mathscr{S}(V) \times v\mathscr{N}(V^*)$ and $(f, w^*) : (A, Y) \to (B, Z)$

$$\Gamma(A, Y) = \{\alpha \in T_V : V\alpha \subseteq A \text{ and } (N_\alpha)^\circ \subseteq \Gamma(Y)\} \quad (5a)$$

$$\Gamma(f, w^*) : \alpha \mapsto (y\alpha)f = y(\alpha f) \quad (5b)$$

where y is given by $y^* = \Gamma(w^*)$; and

$$\Delta(A, Y) = \{\alpha \in T_V : V\alpha \subseteq \Delta(A) \text{ and } (N_\alpha)^\circ \subseteq Y\} \quad (6a)$$

$$\Delta(f, w^*) : \alpha \mapsto (w\alpha)g = w(\alpha g) \quad (6b)$$

where $g = \Delta(f)$.

Cross-connections induced by automorphisms on V

- Recall that an *automorphism* θ of a vectorspace V is an isomorphism from V onto itself.
- For a proper subspace A of V, let $\theta_{\mathscr{S}}(A) : A \mapsto \theta(A)$; and for $f : A \to B$ in $\mathscr{S}(V)$, $\theta_{\mathscr{S}}(f) = \theta^{-1} f \theta$.
- Then $\theta_{\mathscr{S}}$ is a normal category isomorphism on $\mathscr{S}(V)$.
- Similarly, the automorphism $\theta^* : V^* \to V^*$ induces an isomorphism $\theta_{\mathscr{N}}$ on the category $\mathscr{N}(V^*)$ as follows.
- For a proper subspace Y of V^*, $\theta_{\mathscr{N}}(Y) : Y \mapsto \theta^*(Y)$ and for $w^* : Y \to Z$ in $\mathscr{N}(V^*)$, $\theta_{\mathscr{N}}(w^*) = (\theta^*)^{-1} w^* (\theta^*)$.
- By abuse of notation, $(\mathscr{N}(V^*), \mathscr{S}(V); \Gamma_\theta)$ is a cross-connection where
  $\Gamma_\theta : \mathscr{N}(V^*) \to \mathscr{N}(V^*)$ is defined as

$$\Gamma_\theta(Y) \;=\; \theta^{-1}(Y) \text{ and } \Gamma_\theta(w^*) = \theta^{-1}(w^*) \tag{7}$$

Cross-connections induced by automorphisms on V

- And the dual cross-connection $(\mathscr{S}(V), \mathscr{N}(V^*); \Delta_\theta)$ is given by $\Delta_\theta : \mathscr{S}(V) \to \mathscr{S}(V)$

$$\Delta_\theta(A) \;=\; \theta(A) \text{ and } \Delta_\theta(f) = \theta(f) \tag{8}$$

- For the above cross-connection Γ_θ with bi-functors $\Gamma_\theta(-,-)$ and $\Delta_{\theta^{-1}}(-,-)$, the duality χ_{Γ_θ} associated with Γ_θ is given by $\chi_{\Gamma_\theta}(A, Y) : \alpha \mapsto \theta^{-1}\alpha\theta$.
- Then α is linked to β if and only if $\beta = \theta^{-1}\alpha\theta$
- And so

$$\tilde{S}\Gamma_\theta = \;\{\,(\alpha, \theta^{-1}\alpha\theta)\; \text{such that } \alpha \in T_V\}$$

- And hence $\tilde{S}\Gamma_\theta$ is isomorphic to T_V.

- Now in fact, we can show that any cross-connection Γ from $\mathscr{N}(V^*)$ to $\mathscr{S}(V)$ is one of the form Γ_θ for an automorphism θ.
- For the cross-connection Γ with the dual Δ, define

$$(b)\theta = x \text{ such that } \Delta(\langle b\rangle) = \langle x\rangle$$

for all $b \in B$ where B is a basis of V.
- Then θ will be an automorphism on V and then we can show that $\Gamma = \Gamma_\theta$.
- And since the cross-connection semigroup $\tilde{S}\Gamma_\theta$ is isomorphic to T_V, we conclude that every cross-connection semigroup arising from the cross-connections between $\mathscr{N}(V^*)$ and $\mathscr{S}(V)$ is isomorphic to T_V.
- **SO WHAT ?..**

Variant semigroup

- For an arbitary linear transformation θ, let $T_V^\theta = (T_V, *)$ be the variant of linear transformation semigroup with the binary composition $*$ defined as follows.

$$\alpha * \beta = \alpha \cdot \theta \cdot \beta \quad \text{for } \alpha, \beta \in T_V.$$

- Variant of a semigroup was initially studied by Magill(1967) and Hickey(1983); and later by Khan and Lawson(2001), Tsyaputa(2003), Kemprasit(2010), Dolinka and East(2016) etc.
- Then we can see that the cross-connection semigroup Γ_θ described previously refers to the cross-connection arising from the semigroup T_V^θ where θ is an automorphism.
- If we define $\phi : T_V^\theta \to \tilde{S}\Gamma_\theta$ as $\alpha \mapsto (\theta\alpha, \alpha\theta)$, then it can be shown that ϕ is an isomorphism.

Variant of linear transformation semigroup

- Khan and Lawson(2001) had showed that the regular elements of T_V^θ forms a subsemigroup.
- Now if we define categories $\mathscr{S}_1(V)$ and $\mathscr{N}_1(V^*)$ as follows,

$$v\mathscr{S}_1(V) = \{A : A \subseteq (N_\theta)^c\} \text{ and } v\mathscr{N}_1(V^*) = \{A^\circ : N_\theta \subseteq A\}$$

- And imitate the construction as above, we can see that $T\mathscr{S}_1(V)$ is isomorphic to a subsemigroup of T_V and $T\mathscr{N}_1(V^*)$ is isomorphic to a subsemigroup of T_V^{op}.
- In this setting, we have normal cones which are not principal cones.
- If $\mathscr{S}_1(V)$ is *'big'* enough (and that depends on θ), $N^*\mathscr{S}_1(V)$ will be $\mathscr{N}(V^*)$; else a proper sub-category of it.

Variant of linear transformation semigroup

- For $A \in v\mathscr{S}_1(V)$, let $\theta_{\mathscr{S}}(A) : A \mapsto \theta(A)$; and for $f : A \to B$ in $\mathscr{S}_1(V)$, $\theta_{\mathscr{S}}(f) = \theta^{-1}_{|\theta(A)} f\theta$.
- Then $\theta_{\mathscr{S}}$ is a 'proper' *local isomorphism*; and hence a cross-connection.
- Similarly, θ^* induces a dual cross-connection $\theta_{\mathscr{N}}$ on the category $\mathscr{N}_1(V^*)$.
- And the cross-connection semigroup that arises from Γ_θ is the semigroup $\text{Reg}(T_V^\theta)$ of all regular elements in T_V^θ.
- Thus we have a representation of $\text{Reg}(T_V^\theta)$ as a sub-direct product of $T_V \times T_V^{op}$ given by $\alpha \mapsto (\theta\alpha, \alpha\theta)$.
- **This suggests that whenever a complicated ideal structure arises, it is indeed worth taking the risk of 'crossing' into cross-connections !**

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